Fractional and Proportional Reasoning:

Contextualized and Decontextualized

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A student's ability to solve fractional and proportional mathematics problems has been shown to be predictive of success in algebra and higher formal mathematics (Siegler & Pyke, 2013; Lamon, 2007; Bailey, D. H., Hoard, M. K., Nugent, L. and Geary, D. C. 2012; Confrey, 2012). There is not only a supportive link between fractional skill and proportional reasoning (Lamon, 2007), but also evidence that strength in both skills corresponds to higher skill levels in later achievement tests (Siegler & Pyke, 2013)

A decontextualized math problem is one that uses only mathematical symbols to express quantity and structure (Kneebone, 1963). A contextualized math problem is generally considered a problem that is situated in the "real world," most commonly encountered as story problems in math textbooks. But context in a math problem may not only refer to how the problem is situated in the "real-world." Mathematical context can also refer to how a problem is situated within the entirety of a student's mathematical understanding (Pirie, & Kieren, 1994).

To approach mathematics problem solving as either a contextualized or a decontextualized processes does not address the complexity of the relationship between the two contexts. Pirie and Kieren (1994) argued that growth in mathematical understanding is not demonstrated by a single trajectory or path of knowledge within the domain, but can be characterized as categories of knowing. They describe the process of growth as a dynamic restructuring of these categorizes as a learner acquires new knowledge. Formalized understanding occurs as the learner recalls former knowledge and integrates new information. A study of 162 sixth-grade students (Cirino, Tolar, Fuchs, &

Huston-Warren, 2015) found that the integration of decontextualized procedural skills, and the linguistic skills necessary to understand contextualized problems was crucial to student achievement in assessments of fractional skills and procedural reasoning. A bidirectional relationship between decontextualized and contextualized problem solving skills was also found in a longitudinal study of 4th and 5th grade students (Hecht & Vagi, 2010).

Learning trajectories are defined as the likely paths that learners must negotiate to successfully move toward higher-level mathematics skills (Confrey, 2012). Though learning trajectories may describe likely paths to higher-level skills, consensus has not been reached on the pathway that must be traversed. Empson (2011) argues that the fact that multiple studies have not converged on a single trajectory indicates that learning is sensitive to context and cannot be known in advance.

In spite of disagreement over the details, it is generally accepted that the learning trajectory that leads to upper mathematics presupposes procedural skills with fractions and proportional understanding in both contextualized and decontextualized representations (Comfrey, 2012). Procedural knowledge does not only function as a precursor to conceptual fractional knowledge. While contextualized knowledge is demonstrated by high-level practitioners, procedural knowledge remains important, especially to novices (Flyvbjerg, 2006). A related understanding of conceptual knowledge and procedural fluency has been shown to correlate substantially in elementary and middle school students (Cirino, Tolar, Fuchs, & Huston-Warren, 2015; Hecht & Vagi, 2010). Procedural skills contribute significantly to conceptual skills and conceptual fractional knowledge contributes to procedural ability (Bailey, Zhou, Zhang, Cui, Fuchs, Jordan, Gersten, Siegler, 2015). Both approaches are necessary and contribute to true expertise reached by practitioners (Flyvbjerg, 2006).

Purpose of the Study and Research Questions

The purpose of this phenomenological study was to explore the experience of a middle school student, a high school student, and a post-secondary student as they attempted to solve both decontextualized and contextualized math problems involving fractions, ratios, and proportional reasoning.

The following research question served to guide the inquiry and analysis: In what way do participants of different ages make sense of contextualized and decontextualized fractions and proportional reasoning problems?

Participants

The participants were a convenience sample of three siblings who had attended the same schools and been taught by the same sequence of math teachers. The youngest was Katie who was 13 years old and an eighth grade student at a private Catholic elementary school. She was currently enrolled in algebra. The middle sibling was Jon who was 16 years old and an eleventh grade algebra 2 student in a private Catholic high school. The oldest of the three was Chris who was 18 years old and a precalculus student in his first year of undergraduate study. Katie and Chris indicated that they enjoy math. Jon indicated that he did not enjoy math. Participant background information was collected with an online survey before the interviews were conducted.

I was particularly interested in exploring the differences between the siblings experience because of the homogeneity of their home life, the similar trajectory of their educational experiences, and my long relationship with them and their family. The differences in their ages allowed for an exploration of the possible divergence of their experience.

Research Design and Methodology

The methodological framework guiding this study is interpretive phenomenological analysis (IPA) (Smith, Flowers, & Larkin, 2009). This qualitative research approach incorporates phenomenology, the study of lived experience; hermeneutics, the theory of interpretation; and ideography, the experience of each individual participant. Consistent with IPA studies, this study has a small sample size that is reasonably homogenous that allowed me to examine convergence and divergence in some detail. Ordinary everyday experience becomes an experience of importance as a person tries to make sense of the significance through reflection (Smith et al., 2009). In order to record the participants' active meaning making of their experience as they solved the problem set, a talk-aloud protocol was used. In addition I engaged each of them in a reflective interview conversation following completion of the problem set.

Site Selection

The interviews took place in the office where I regularly meet with the participants as their teacher in another domain. The space is lit with natural sunlight from three large windows along one wall. The room contains a grand piano, a tall bookcase and a double-sided desk. For each interview the participant and I sat on opposite sides of the desk allowing me to observe their processes as they solved the problems. I recorded their responses as they worked with a digital recording device. The follow-up interviews were conducted in the same space and also recorded.

The orientation of working on either side of the desk was common to the participants and to me, creating a familiar environment for both of us. The atmosphere remained relaxed as the participants worked through the problem set. The space was chosen because it was convenient, and also because it was a space where the participants were known to be comfortable. The issue of hegemony was addressed by explaining to the participants in detail the difference between my normal role as their teacher and my role as researcher for this study.

Data Collection

Primary and secondary sources provided the data that was used for the analysis of this study. Because the aim of this study was to understand the experiences of the participants as they worked through the problem set, primary data sources consisted of the completed problem set sheets, a talk-aloud interview conducted while the participants completed the problems set, and a follow-up one-on-one interview. Secondary data sources consisted of field notes from observations of the participants as they worked, and sound recordings and transcripts of the talk-aloud and follow-up interviews.

The problem set sheets (Appendix) completed by the participants consisted of a sheet of four decontextualized problems, and a sheet of four problems that contextualized the mathematic concepts presented in the decontextualized problems. The sheets present similar math problems in both contexts, allowing participants to apply the same mathematic skills to both decontextualized and contextualized problems. The difficulty level of the problems is at the fifth and sixth grade standards for fractions and proportional reasoning as described in the Common Core State Standards (National

Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

I asked clarifying questions when participants' thought processes were not evident in their mathematical procedures or their verbal narrative. Following each interview, I recorded my initial thoughts and impressions in field notes.

Member checking was achieved through follow-up interviews with participants to clarify my perception of the participants' approaches to problem solving and to elicit more detail about their experience.

Data Analysis

I conducted data analysis using the IPA framework as a guide (Smith et al., 2009). I listened to the audio recordings and reviewed the transcripts and my field notes holistically at first. A second review of the recordings and other data allowed me to bracket my initial impressions using analytic memo writing.

Initial notating consisted of a close textual analysis of the data. I assigned codes in three forms: response to presented context, procedural response with explanation, and procedural response with no explanation. Analytic memo writing continued at this stage

As the data, memos, and codes were analyzed emergent themes developed. At this stage I maintained my focus on both the differences and the similarities between the participants' responses. This allowed me to identify patterns and make connections across the emergent themes.

Role of the Researcher

I have known the participants for most of their lives, as their music teacher throughout their grade school years, and as a family friend. As a result I have a close, complicated relationship with the participants.

The nature of our interactions within the music studio is primarily collaborative. Music lessons are conducted in a one-to-one setting that allows the students to freely discuss their frustrations and challenges. As a result, our working relationship is comfortable. We are used to communicating with candor, and trust is inherent.

The challenges that my close relationship to the participants present include the natural power distance that exists between us, both as their music teacher and as a friend of their parents. In an effort to bridge the distance I described my role as participant-observer to the participants. Points of departure from our normal working relationship included the significant difference between how we normally interact and how the observations and interviews were conducted. Because my working relationship with the participants existed in another domain, the research experience was a novel one for all of us.

My familiarity with each of their learning styles made it especially important for me to allow each of the participants to explicitly state their thoughts and describe their experiences during the process so that their perceptions are foregrounded in the findings.

Findings

Though the participants were free to solve the problems in whatever order they wished, all worked through the problem set in the numbered sequence given on the sheets. Consequently, they each completed decontextualized problems before they approached the contextualized word problems.

Katie relied heavily on her procedural knowledge solving both the decontextualized and contextualized problems. As she was working through the decontextualized problems I asked her several times why she used the methods she was using. She consistently answered, “It’s how I was taught to do it.” When asked to consider how she would check to see if her answers made sense she responded, “They taught us to plug it back into the problem.” Her approach to checking her accuracy was consistently decontextualized. In all cases she relied upon outside experts to support her approach to solving problems stating at one point, “It’s how the book taught me.” Relying upon her procedural skills, Katie did not attempt to conceive of the problems in more generalized or more concrete ways.

When Katie turned to the sheet of word problems she stated that she was “kind of hesitant to do them because I have to read.” Her approach to solving the contextualized word problems was to convert them into decontextualized equations. She then solves them using the same processes she had used on the decontextualized problems. When asked about her approach she said, “Once I started the word problems they became plain equations.”

In a follow-up interview Katie described how she viewed the problems in order to solve them. Approaching both the decontextualized problems and the word problems she had converted into equations, she explained that she viewed the fractional amounts as parts of circles. In her words, “If you draw a circle and divide into thirds and another circle and divide into fourths, the third is going to be greater than fourth.” In spite of the fact that her work appeared to be purely procedural, in order for Katie to derive meaning from the problems she contextualized the fractions as part-to-whole relationships.

Jon appeared to approach the decontextualized problems in the same procedural way that Katie did. He followed accepted order-of-operations methods to complete the problems. Differences in Jon and Katie's approaches emerged when I asked him how he would check to see if his answers made sense. Looking at the fractions he said "I visualize it as a pizza." In order derive meaning, he contextualized the problems that had been presented in symbolic form. In a similar way he checked his answers by estimating the quantities of resulting fractions as "units." When he a solved a problem that resulted in an improper fraction he was unable to assess his accuracy because he could not successfully reduce the answer commenting, "My method might be wrong, I don't really know because I don't know why it works." When he could not contextualize an answer, he checked the accuracy of his work by estimating the given amounts, and rounded his answers to see if they met his expectations saying, "I think this is close enough." Like Kate, he would often plug his answers back into the original problems. While Katie carefully reworked the problems to test her answers, Jon would look over the problem with his answer plugged into it and consider whether the numbers made sense. In most cases he accepted the quality of his work and moved on without reworking the problem.

When Jon turned to the page with the contextualized problems, he clenched both hands into tight fists on the desk and loudly sighed, "Word problems." Asked about the reaction, Jon said that he a "lot of bad experiences with word problems popped back up." The first contextualized problem he appeared to decontextualize by converting it into an equation using a procedure similar to the one that Katie used. As he worked through the problem it was apparent that as he wrote numbers he still conceived of the problem in the context in which it was presented saying, "Two days and we need to get to five, so that's on one side of the equation. When you add these three days up that should be five." The following problems he also kept in context stating, "To make it easier since I'm kind of a visual guy, I'd make myself a little tabletop" for the proportional problem, and "I would want to know how many dollars is in a dinar," for the currency conversion problem. As he drew a rectangle for the farmer's fence problem he commented, "you need a thousand feet of fence that's wrapped around it." For all of the contextualized problems, he solved them within the context in which they were presented.

In the follow-up interview I asked Jon about the difference between his initial reaction to the sheet of word problems, and his apparent state as he worked on them. He displayed no apparent anxiety once he started solving the contextualized problem His voice inflection and body posture appeared quite relaxed and he conversed easily about his approaches. Jon said, "To be honest, these weren't as hard as I thought they would be. They were not all that complex."

As Chris worked on the sheet of decontextualized problems, he appeared confident in his procedural knowledge, displaying very little hesitation. His comments indicated that his approach was in fact purely procedural, staying within the abstract domain: "Multiply both sides by two first, divide both by seven," "Because that way, both sides are positive." Chris relied on intuition rather than recalculating as Katie had done, or contextualizing the problems in the way that Jon did. When I asked him how he knew his answers were correct he responded, "It makes more sense," and "On this one it just feels right, so I just move on."

Chris comfortably moved between contextualized approaches and decontextualized procedures as he worked through the sheet of word problems. When approaching the first in the set, like Katie and Jon, Chris converted the problem into an equation. Unlike Jon, who maintained a contextualized image of the problem as he worked, Chris appeared to approach his resulting equation procedurally saying, "I would make that one big fraction." He described his final answer within the context of the word problem saying, "You can look at how far she has walked, when you add those up you already know she's walked a little over a mile so that makes sense. Just under a mile left. It's almost a whole mile."

Chris' approach was not consistent across all of the word problems though. He used the same decontextualize/re-contextualize process for the proportional table and farmer's fence problems that he had used with the first contextualized problem. But when faced with the monetary exchange problem, he maintained a consistently contextualized perspective that was similar to the way that had Jon approached the problem. This was clearly evidenced by the nature of his comments as he worked: "I would divide twenty two by two because each two is three dollars, so I would have eleven pairs of dinars." He based his conversion on "how many one dinars there are to each dollar, or how many groups of five there are."

I asked Chris about his initial reaction to the sheets of problems during a follow-up interview. He said that the decontextualized problems were "harder than I remember. I haven't done these in a while." He said that he first thought "oh no" when he saw the sheet of word problems. Explaining his procedural approach to the contextualized problems he said, "I was awful at word problems, but these were not terrible. I try to approach it mathematically. I try to pull out the math and get rid as much as possible." In explaining how he re-contextualized the problems as he said, "I more visualize what they're trying to portray. I saw a fence around a field first. I saw people with money." Consistent with his response to the survey he stated that "(I) still like doing math."

Discussion

It is certain that Katie, Jon, and Chris will have different experiences as they make their way through their formal mathematics training. Additionally, they have different attitudes and levels of aptitude for the work. Still, there are some salient commonalities between them. They have been raised in the same household, and have attended the same schools. Their relationships with me differ in the way that all relationships differ, but are the similar in the quality and quantity of our interactions.

All three of the participants relied on their procedural skills to solve the problem set. Katie was completely dependent upon her procedural abilities to solve all of the problems. She is only a couple of years beyond the grade levels in which she would have been introduced to procedures for solving fractional and proportional problems. The approaches she used would be expected at this point in her mathematical development. The recent attainment of her skill set was also a possible explanation for why she visualized fractions as part-to-whole relationships. These relationships are commonly used to present early level concepts of fractions to grade school students and are part of the Common Core State Standards for third grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Jon and Chris were able to remain within the contextual domain of several of the problems as they solved them. Jon attempted to create a contextualized model as he worked on all of the problems: contextualized and decontextualized. Problems that did not allow him create context left him dependent upon his procedural skills, leaving him unsure of his answers and unsatisfied with the methods he used to check accuracy. Though his procedural skills were mostly correct in their application, he was not comfortable moving between contextualized and decontextualized approaches to problem solving

Chris, on the other hand, displayed sound procedural skill and comfortably switched between contexts, sometimes while working a single problem. In spite of his initial hesitation approaching the decontextualized sheet of problems, he was able to solve them efficiently.

Pirie and Kieren's (1994) model of the development of mathematical understanding as categories of knowing that are restructured as a learner acquires new knowledge may possibly explain some of the differences in the participant's approaches to solving the fractional and proportional problems. Chris had the most experience of the participants, and the integration of the categories that Pirie and Kieren describe fit the way that he easily moved between the categories of contextualized and decontextualized problems. Jon was able to maintain a contextualized basis for his problem solving, but was not able to integrate his procedural knowledge into his models. Katie, who was at the beginning of her formal mathematic training, was still dependent upon "how we were taught."

Further research that would replicate this study with the same participants when they have progressed further in their mathematic development would allow for a more coherent understanding of the development of their fractional skills and procedural reasoning.

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Appendix

Problem Set

1.

2.

3. Solve for x:

4. Solve for n:

5. Clara's goal is to walk 5 miles in a week. On day one she walked 1 ¾ miles. On day two she walked 2 1/3 miles. How many more miles will Clara need to walk to meet her goal.

6. A diagram of a tabletop is 3 inches by 6.5 inches. The scale factor is 1:12. How large is the actual tabletop

7. The money used in Jordan is called the Dinar. The exchange rate is $3 to 2 dinars? How many dollars would you receive if you exchanged 22 Dinars? What if the exchange rate is changed to $3 to 5 Dinars?

8. A farmer has 1000 feet of fence to fence the field. He wants the field to be twice as long as it is wide. What are the dimensions of the field?