Courtney Baker: Check APA for how to do a running head.

A Comparison of Fractional and Proportional Reasoning Skills:

Clinical Interviews of Three Siblings

Kimberlie Fair

George Mason University

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In February and March of 2016 I conducted a series of clinical interviews with three participants who were in middle school, high school, and undergraduate study. The purpose of the clinical interviews was to explore the participants’ understanding of rational numbers and proportional reasoning as they solved a set of mathematical problems. I chose the topic because research indicates that a working knowledge of fractions, rational numbers, and proportional reasoning is necessary for success in algebra and beyond.

Courtney Baker: What research? Cite it.

Literature review

Procedural fluency with fractions and proportional reasoning skills are critical to the success of learning algebra and higher formal mathematics (Bailey, Hoard, Nugent, & Geary, 2012; Confrey, 2012; Lamon, 2007; Siegler & Pyke, 2013). The National Mathematics Advisory Panel (2008) found that fractional knowledge is foundational for algebra, that conceptual knowledge impacts problem solving performance, and that proportional reasoning abilities predict mathematical performance outcomes.

The concept of learning trajectories presents a coherent model for understanding the relationship between fractions and success in higher mathematics. Learning trajectories are the likely paths that learners must negotiate to successfully move toward higher cognitive reasoning (Confrey, 2012). Empson (2011) refines the concept by pointing out that the learning trajectory of an individual cannot be known in advance, and that there are indications that trajectories may be context-dependent. He argues that this

Courtney Baker: which concept? I think you mean learning trajectories.

dependency may explain the fact that multiple studies have not converged on a single trajectory.

Learning trajectories lead to upper mathematics presuppose procedural skills with fractions and proportional understanding (Confrey, 2012). The relationship between contextualized and decontextualized mathematical skills is complex, consisting of multiple approaches and viewpoints. While it is generally accepted that practitioners at the highest levels demonstrate contextualized knowledge, procedural knowledge is important, especially to novices (Flyvbjerg, 2006). The purpose of procedural knowledge attainment is not only usefulness as a precursor to the understanding of conceptual fractional concepts. Several studies have found that procedural skills contribute significantly to conceptual skills and conceptual fractional knowledge contributes to procedural ability (Bailey, Zhou, Zhang, Cui, Fuchs, Jordan, Gersten, Siegler, 2015). Both approaches are necessary and contribute to true expertise reached by practitioners (Flyvbjerg, 2006).

Courtney Baker: This sentence is awkward. I think it is the word usefulness, but the way it is written it leaves the author hanging.

Courtney Baker: Here you say several – yet cite one.

Participants

The participants were three siblings who have attended the same schools, and have thus far been taught by the same sequence of math teachers. The youngest is Katie who is 13 years old and is an eighth grade student at a private Catholic elementary school. She is currently enrolled in algebra. The middle sibling is Jon who is 16 years old and is an eleventh grade algebra 2 student at a private Catholic high school. Chris is the oldest of the three and is a pre-calculus student in his first year at a community college. Survey responses from Katie and Chris indicated that they enjoy mathematics. Jon indicated that he does not enjoy mathematics.

Problem set

The problem set includes five decontextualized problems, and five contextualized word prob . The purpose of the design was to give the participants an opportunity to demonstrate both their procedural skills and their concep abilities. While a related understanding of conceptual knowledge and procedural fluency has been shown to correlate substantially in a longitudinal study of 4th and 5th grade students (Hecht & V 2010), I was curious to see if this correlation would manifest in older participants.

Observations and Interviews

All three participants demonstrated procedural fluency in the decontextualized problems. All participants chose to begin with problem 1. They all began by eliminating the parenthesis. Chris removed the parentheses by distributing the negative, and then changed the fractions into equivalent fractions with common denominators. When asked why, he said that that is how he had always done it, and could not think of why it works. Katie and Jon added the fractions within the parenthesis first then subtracted the result from the whole number. At this point their processes diverged. Katie borrowed one from the whole number and subtracted a mixed numeral from the newly created mixed number. Jon, like Chris, converted the whole number to an equivalent fraction with a common denominator. Jon converted his improper fraction answer to a mixed numeral. Katie’s final answer was already in mix numeral form because of her approach. Chris left his fraction in improper form. All participants answered the question correctly.

All three participants solved problem 2 using the -and-multiply method, initially obtaining the correct answer. In attempting to reduce his answer Chris made a mathema error. He rejected the result along with his initial correct answer. He

Courtney Baker: Refer to your clinical interview protocol at this point i.e. (see Appendix A)

Courtney Baker: A great balance.

Courtney Baker: When you link this research to the design of your clinical interview you are strengthening your case and the choice of your questioning.

Courtney Baker: The first few sentences all have a similar start. Try to incorporate varied sentence structures – especially with sequential sentences and in the beginning of a section.

Courtney Baker: NO! :/

Courtney Baker: What was the error that he made?

returned to the original problem and implemented a cross cancellation technique, ac[redacted]ing the incorrect answer that resulted from the flawed procedure. All three participants were asked about the use of the flip-and-multiply technique. Each was confident that the method was sound, but was unable to conceptualize the problem in a way that would allow them to check their results for accurac[redacted]atie and Jon correctly accepted their answers as correct.

Problem 3, a missing value problem, was approached differently by each of the participants. Chris chose to cross multiply. He initially calculated the value for x correctly, but then continuing to cross-multiply he added a coefficient to x which he then removed by reversing the step he had just completed. He seemed unaware of the reversal he had just carried out. When checking his answer he again used the flip-and-multiply technique after writing the compound fraction in sentence form. He chose to keep the answer as an improper fraction. Katie multiplied both sides by the denominator under the missing value x, then divided her result by the denominator of the other side. In working through the division portion of the problem she was stumped by the long string of integers to the right of decimal. When asked if the question made sense, she rounded her answer and estimated the value each of the numerators divided by the denominators and came to the conclusion that she had made an error. She repeated her procedure and came up with an answer that satisfied her. Rounding to the hundredth place, she indicated that it is a common practice in her current class. Her answer, as far as she took it, was correct. Jon approached the problem by turning the fraction with the unknown number into an equivalent fraction with the same denominator as the given fraction. He removed the denominator, indicating that this can be done because both sides can be multiplied to

Courtney Baker: Interesting that he did not accept it at first, but then was later okay with it.

Courtney Baker: Asking "Does that make sense?" is such an important piece of truly understanding as you mentioned in class last week.

remove it. He then proceeded to divide by the decimal number coefficient that he had added to the unknown to isolate the *x*. Like Katie, he ended up with a decimal number that he rounded at the thousandths place. He was sure that if he continued he would find the complete answer, but chose to stop at 3 decimal places. He used the same estimating technique that Katie had used to check his answer and was satisfied that it was correct. He had made a mathematical error in the hundredth place. Even though their approaches were different, all three demonstrated commonly used procedures to solve the problem. They used different, but mathematically correct methods to check their answers.

The 4th decontextualized problem placed an unknown *n* on both sides of the equal sign in a fractional proportional equation. Chris began by attempting to cross multiply, but then decided that he only needed to multiply the right side to make it an equivalent fraction with the same denominator as the left side. Using algebraic methods, Katie first multiplied both sides by the denominator of the rights side to eliminate it. She then repeated the process on the left side. Jon used Chris' method of similar denominators without hesitation. All three participants obtained the correct answer using algebraic fundamentals. All three checked their answers by plugging the answers back into the original equation and were satisfied that they were correct.

The 5th problem was the first of the word problems. It was designed to replicate the first problem in contextual form. Jon approached the problem by creating an equation that, if carried out, should have produced the correct answer. He made a conversion error from mixed numeral to improper fraction, resulting in an incorrect answer. When asked if the answer made sense he incorrectly reviewed the given problem in an effort to support his answer. Katie chose to add the fractional elements first and then subtracted her answer from the five-mile goal. Here, as in the first problem, she converted her improper fraction into a mixed numeral, then borrowed from the integer in order to carry out the subtraction. When she checked her answer she converted the given fractions, her answer, and the goal into fractions with a common denominator. This simplified correctly. She then decided that it must be correct because if she thought of the given fractions as parts of a circle, then the answer must be less than one. Chris used a procedure that replicated Katie’s, but put all of the numerals into fractions with common denominators to solve. He described checking his answer in the same way that Katie did, but talked about it in terms of number magnitude to determine that the answer must be less than one, rather than Katie’s method of using circles. Chris is comfortable with the abstract values of fractional numbers and how they relate. Katie, on the other hand, thinks of fractions as parts of a whole (circle) in order to visualize the problem. Jon is comfortable with his procedures supporting his incorrect answer by a misreading the problem when asked to check.

Chris and Jon both approached question 6, scale factor of a table diagram, by drawing the diagram first. Using scale notation, Chris multiplied to get the correct dimensions of the table. When asked how he saw the relationships, he explained that he saw them as proportions, and then wrote the relationships in proportional, equivalent fraction form. Jon explained that he is visual and has experience with models, so to solve the problem he needed to see a little tabletop. He approached the problem by visualizing the real table, explained that the dimensions had to be divided to end up with the diagram dimensions, so he needed to multiply to find the tabletop dimensions. He is unaware of the recursive nature of his approach. He used simple multiplication to solve the problem.

Courtney Baker: Definite conceptual understanding.

Katie does not draw the diagram, but proceeds directly to multiplying the diagram measurements to find the tabletop dimensions. All answers are correct.

Question 7 is a monetary conversion problem. Katie set the problem up as a missing value proportion and solved it by cross-multiplying and dividing in the same way that she solved the missing number problem. When asked to recalculate with a different exchange rate she set up the problem with the new numbers and correctly solved the new problem. Chris approached the problem by thinking in units of dollars equivalent to a number of dinars. Initially this approach worked well, but when asked to find an answer with a different exchange rate he was stumped. After a thoughtful pause he followed the same method he used before resulting in the correct answer. He indicated that the problem was more difficult with the second set of numbers. Jon chose to standardize dollars in terms of one dinar and then multiplied to get the correct answer. He then rounded his answer to an incorrect answer. When I asked him why he changed it he said because the new answer was correct. When given the different exchange rate he used the same method he had used previously and calculated the correct answer. Neither Jon nor Chris associated the problem with the missing value problem.

Question 8 is a proportional problem, which all three participants set up with a diagram. Both Katie and Chris created an algebraic equation that represented the problem, then solved for the unknown. Jon set up his diagram in terms of width and determined that the perimeter was equal to 6 widths. To find the width value he needed to divide the length of fence by 6. When faced with the repeating decimal results of the division, Chris recognized the fractional equivalent and was comfortable giving the farmer correct measurements in fractional form that he felt would be useful to the farmer.

Both Katie and Jon were dissatisfied with the repeating decimal answer. Jon's solution was to round the answer up to the next integer and incorrectly adding the length and width, resulting in an answer that he felt was correct. After rounding up, Katie correctly concluded tha[illegible] answer was too large though she did not know why. She carried the decimals further, and determined that while the new answer was closer, it was not satisfactory. I prompted her to think about how the answer might be given as a fraction instea[illegible]a decimal. Her response was that she was more comfortable with decimals and found them easier than fractions.

Courtney Baker: thinking about the reasonableness of the answer...

Courtney Baker: A great follow up question.

When asked, all three participants indicated that they had been taught to convert fractional answers into decimal equivalents. Chris explained that in his high school pre-calculus course he was instructed to leave his answers in fractional form. This was also true of his current math course and had become a habit for him. Except for the currency exchange problem he was comfortable with fraction forms for answers. In the case of the farmer's fence problem, he had readily converted the repeating decimal to its fractional equivalent. Katie and Jon left answers that were less than one in fractional form, but converted all proportional problems into decimal forms. When presented with addition and subtraction of fractions in problems 1 and 5 they left their answers in mixed numeral form. Once Jon had rounded his answer to the farmer's fence problem he again miscalculated when his estimated to support his incorrect answer. Katie felt that her answer was [illegible]mplete, but gave up searching for a solution that would work in the real world.

Courtney Baker: I wonder if this is due to her perseverance (or lack there of) or a time constraint (perceived or actual).

Relationship to Literature

Recent research on learning trajectories (Confrey, 2012) indicate that it would be reasonable to assume that all three of the participants would have a good working knowledge of fractions and proportions at this point in their studies. Both Chris and Jon have successfully completed algebra, and are working higher-level concepts. Katie is currently successfully working her way through algebra.

Studies have found that procedural skills and conceptual knowledge have a reciprocal relationship: strengthening one appears to strengthen the other (Flyvbjerg, 2006). This phenomenon was not demonstrated by the participants. While Chris is the most advanced of the three participants, he was the least likely of the three to think beyond familiar procedures in the decontextualized problems and in many cases, could not offer reasons for the procedures he was carrying out. Assuming his preparation was the same as his younger siblings, it seemed that his higher-level work and separation from simple fractional problems may have contributed to his discomfort. His procedural skills while working in the contextualized problems was solid and he did not display the same discomfort. Jon expressed a preference for decontextualized problems, but was equally successful solving both types of problems. Katie was comfortable in both forms. She explained that when she reads a story problem she sees it as an equation, and once it is in equation form she is comfortable solving it. Her skills supported her problem solving in the interview, except for her inability to convert her answer to the farmer's fence problem into a form that is appropriate for the context.

Courtney Baker: Plural 'studies' here – yet only one is cited.

Courtney Baker: advanced how – by age, math reasoning, level, ability...be specific to help your reader come to the same conclusions that you are coming to.

While multiplicative explanations to proportional reasoning problems do not present a clear perspective on a student's' understanding of proportionality, Lamon (2012) includes the ability to discern a multiplicative relationship between two quantities, and the ability to extend the relationship to other pairs as a demonstration of proportional reasoning. Jon did not use normally expected algorithms when solving the proportional word problems but was able to discern the relationships described by Lamon, specifically in his ability to extend the relationship in the currency exchange problem. In all cases his reasoning was sound and his answers were correct when they were not marred by arithmetic errors, indicating an ability to reason proportionally.

Further Questions

There is fertile ground here for further exploration of fractional concepts and proportional reasoning by students of different ages and mathematical attainment. While conceptual knowledge was demonstrated with mathematical problems were embedded in a context, conversation with the participants that was beyond the scope of this paper indicated that there was a great variety in the ways that mathematical concepts informed the procedures that the participants used. The participants' understanding of why procedural methods worked or did not work seemed to be correlated with how well they solved problems. In conversation with the participants, it was clear that the way they conceived of fractional and proportional problems varied greatly from each other and from problem to problem.

In the pre-survey Jon indicated that he did not enjoy mathematics. Is his non-standard approach to problem solving a hindrance to [redacted]uccess? Theoretically he demonstrates an understanding of proportional reasoning. He approached the story problems with a positive attitude in spite of early misgivings and clearly enjoyed the process. Can allowing him to articulate his understanding have an affect on his attitude? In the larger context, in what way would giving voice to students affect their outcomes[redacted]

Courtney Baker: If only there was a follow-up interview!!

Courtney Baker: Try not to end with a question.

References

Bailey, D. H., Hoard, M. K., Nugent, L. and Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of experimental child psychology 113*(3). doi:10.1016/j.jecp.2012.06.004

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., Gersten, R., & Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology, 129,* 68–83. doi:10.1016/j.jecp.2014.08.006

Confrey, J. (2012). Better measurement of higher cognitive processes through learning trajectories and diagnostic assessments in mathematics: The challenge in adolescence. In V. F. Reyna, S. B. Chapman, M. R. Dougherty, & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making*. (pp. 155-182). Washington, DC: American Psychological Association. http://dx.doi.org/10.1037/13493-006

Empson, S.B. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast 8*(3).

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry*, *12*(2), 219-245.

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology, 102*, 843–859. http://dx.doi.org/10.1037/a0019824

Lamon, S. J. (2007). Rational numbers and proportional reasoning. F. Lester, Jr.(Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.

National Mathematics Advisory Panel. (2008) *National Mathematics Advisory Panel Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology, 49*(10), 1994-2004. doi:10.1037/a0031200

Appendix

Clinical Interview Problem Set

1. $5 - \left(\frac{1}{3} + \frac{5}{2}\right) =$

2. $\frac{2}{3} \div \frac{3}{4} =$

3. Solve for *x*: $\frac{45}{7} = \frac{x}{2}$

4. Solve for *n*: $\frac{3n-7}{6} = \frac{2n-5}{3}$

5. Clara's goal is to walk 5 miles in a week. On day one she walked 1 ¾ miles. On day two she walked 2 1/3 miles. How many more miles will Clara need to walk to meet her goal.

6. A diagram of a tabletop is 3 inches by 6.5 inches. The scale factor is 1:12. How large is the actual tabletop

7. The money used in Jordan is called the Dinar. The exchange rate is $3 to 2 dinars? How many dollars would you receive if you exchanged 22 Dinars? What if the exchange rate is changed to $3 to 5 Dinars?

8. A farmer has 1000 feet of fence to fence the field. He wants the field to be twice as long as it is wide. What are the dimensions of the field?