Teaching and Learning

In a

Physical and Digital World

Kimberlie Fair

George Mason University

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The rise of digital technologies has made it normal and necessary for learners and educators to navigate between and within both the physical and the digital worlds. Communication is freed and bound by the devices we choose to employ. On screens large and small we consume and create. In spite of the changes that digital technologies have wrought, we remain physical beings that continuously react and adapt to the physical world. Though current changes are new to us, the challenges that new technology brings have been a constant in education.

It is normal to cling to methods that are familiar and useful, and the challenges of new technologies by their very nature disrupt the familiar. While the ancient Greeks developed algorithmic techniques as reliable and consistent method for solving mathematical problems, the reaction from many was a fear that being able to solve complex mathematics problems with less mental work would result in a loss of mental power (Usiskin, 1998). What we know now is that rather than slowing or halting the intellectual development of the world, the ease of calculation that algorithms provide allowed for the development of increasingly more complex and sophisticated mathematical understanding and discovery (Cooke, 2011).

In spite of the early development of algorithmic techniques, the means to deploy written mathematical techniques were limited by the availability of writing materials. It took the ingenuity of the industrial revolution to make writing tools and materials cheap and easily accessible (Petroski, 1990). By the end of the 19th century paper and pencil were becoming standard tools for communicating mathematics practice and conducting assessments in education.

The technology of paper and pencil dominated much of the 20th century and was only challenged by new technologies in the last 30 years. Though we still do not have a clear picture of the impact of technology on educational outcomes and researchers continue to search for ways to identify and measure effects, the education community continues to push for integration. Perhaps it is the realization that digital technology is already fully integrated into our lives and therefore must be also used for education. The use of technology is present in both recommendations for instructional practice and developed standards. The National Council on the Teaching of Mathematics (2014) states as a foundational principle “an excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking.” (p. 5). The meaningful use of technology is interwoven throughout the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) for mathematics specifically for high school students. Technology is defined as a tool in the introduction to standards for mathematical practice used to mindfully work on mathematics and as a tool to strengthen mathematical practices. Technology use is not mentioned in the lower grades standards.

There is always a concurrent push away from, and a pull toward new technologies in education. What does it mean to use technology as a tool to strengthen mathematical practices? The constant search for fresh approaches and new perspectives leads educators to reach for the newest object of desire. But once the object is in hand, the preconceived ideas of how the object will be used and what the results will be seldom match the reality. The presence of new technology does not ensure its use and it is not uncommon for educators to be challenged to integrate technology into their practice when it is readily available (Cuban, 2001). The long association of physical writing tools with the practice of mathematics has resulted in a large body of research that compares emerging technologies with paper-and-pencil techniques.

Calculators

The first real digital challenge to paper-and-pencil dominance came in the mid 1980s in the form of graphing calculators. An article (Steen, 1987) that embodied the optimism of the time argued that the use of calculators in the post-secondary classroom would prepare students to use the tools that professionals in the working world were already using. The author characterized procedural techniques currently in use in post-secondary classrooms as tools that merely allowed students to get by. Her stand was in opposition to her perception of mathematics educators' assertions that graphing calculators were inappropriate for instruction because they did not allow for mental insight and abstract calculation. She believed that digital technology would democratize college mathematics, obliterating the impact that weaknesses in algebra skills had on students' ability to successfully pursue college level mathematics study. She predicted that by the 1990s calculators would replace template exercises and "mimicry mathematics" The result would be a shift toward exploration and insight and intuition based mathematical methods. Ultimately she declared, "The era of paper and pencil mathematics is over."

While Steen's declaration proved to be premature, the exploration and insight that she predicted graphing calculators would provide was demonstrated in contemporary research. The view that calculators were necessary and useful seems to have been fairly universal among educational institutions at the time. The National Council of Teachers of Mathematics curriculum and evaluation standards (1989) called for graphing calculators to be available to all students at all times. An early review of research on calculators in the classroom (Dunham & Dick, 1994), found that experimental research on secondary and post-secondary students graphic calculator use had mixed results. The researchers found that all of the experimental studies not only provided calculators to their treatment groups, they also gave different curriculum and instruction to the treatment groups than the curriculum and instruction they gave to the control groups. Durham and Dick determined that it was both the combination of technology and the resultant changes in curriculum and instruction that drove the differences in outcomes. They determined that isolating the impact of the calculators on student achievement was impossible in the studies they reviewed. The conclusions they were able to draw were that calculators provided more free time for instruction, provided more tools for students who had weaker algebra skills, and that by freeing students from numerical computation, students were able to concentrate more on analyzing problems. Questions that remained included whether generating graphs with graphing calculators allowed students to confirm results achieved by paper-and-pencil techniques and whether the quality of the graphs generated had an impact on participants' results. They also recommended further research on the question of whether or not the graphing calculators encouraged exploration and instruction. Across the studies reviewed, participants who used the calculators did display more flexible problem solving approaches and were more willing to engage in problem-solving tasks for a longer period of time.

Subsequent research that sought to address many of the questions raised by Dunham and Dick about how students use graphing calculators for problem solving. A 1996 study (Drijvers & Doorman, 1996) of pre university students who were familiar with graphics calculators found that graphing calculators facilitated participants' ability to understand the mechanics of global graphic representation and the graphical conceptualization of problems. The calculators allowed students to check their work quickly and investigate how graphs were created. The use of the graphing calculators was also seen as a tool that encouraged students to reflect on their work.

Calculators allowed the participants' emphasis to move from rigid procedures to flexible problem-solving techniques.

A study in 2011 sought to determine why technology had not by then impacted the mathematics classroom as expected (Hitt, 2011). The researcher found that teachers generally fell into two camps: either they wanted to use graphing calculators without worrying about concepts, or they rejected the use of graphing calculators because they feared that their use would inhibit the development of students' conceptual mathematics understanding. The researcher predicted that over time the use of electronic tablets and interactive whiteboards would tip the balance in favor of digital technologies. The report included a qualitative study of secondary in-service teachers (Hitt, 2011) that found that a balance of paper and pencil techniques and the use of technology had the greatest impact on learner outcomes. The study also found that mathematicians and teachers had higher skill levels that allowed them to predict graphing results that they get using technology. Students who had not achieved high levels of competency tend to blindly believe results provided by a calculator without assimilating them. The study found that collaborative learning was a way to counteract the trend. Allowing students to work together to conjecture outcomes gave them the environment necessary to develop schema that then could be used with large numbers. The researcher argued that the use of graphing calculators in a collaborative environment would allow for greater student achievement and might impact integration of technology into the classroom.

More recently, an exploratory study (Kop, Janssen, Drjvers, Veenman, & van Driel, 2015) attempting to identify efficient strategies for algebra students to graph formulas found that experts used qualitative reasoning strategies to visualize graphs. Five experts and three secondary-school math teachers were asked to solve complex graphing tasks using graphing calculators and pencil-and-paper techniques. The study found that using graphing calculators resulted in more accurate, heuristic approaches. Pencil-and-paper techniques tended to incorporate mostly quantitative reasoning. The researchers determined that novices needed to learn to reason about graphs using pencil and paper first in order to use technology effectively. Paper-and-pencil techniques were more effective for establishing connections. Experts demonstrated the ability to work efficiently between and within representations. While the global approach to representation that graphing calculators presented were seen as more powerful, the point wise approach employed with pencil-and-paper was needed for novices to develop the process/object perspective necessary for efficient context switching and the development of the cognitive schemas that experts possess.

It appears that finally the paper-and-pencil vs. calculator debate has turned into a conversation about the appropriate use of the technology in the classroom. While the use of graphing calculators is mandated in most formal mathematics classrooms, the need for solid paper-and-pencil skills still exists. Though several universities require graphing calculators in their calculus classes, their use is barred from tests and assessments (Morris, 2014; Syllabus, 2016; Clemson, 2016).

Computers for Assessment

In some ways the argument over the appropriateness of the use of graphing calculators in mathematics education seems quaint when viewed against the explosion of the use of computers in education. The impact of computers on instruction, learning, and assessment has been studied extensively and results are decidedly mixed.

Several studies have found that delivery of assessments by computers had a positive effect on participants. A matched-pair experimental study of 260 ten-year-olds (Hargreaves, Shorrocks-Taylor, & Swinnerton, 2004) found that computer delivery of tests had an overall positive effect when compared to identical tests delivered with paper-and-pencil. A 2009 review of the literature (Macedo-Rouet, Ney, Charles, & Lallich-Boidin, 2009) found that for undergraduates, web delivery of assessments resulted in better outcomes for open-ended questions, and that there were greater benefits to high achieving students.

Conversely, a 2008 national study (Bennett et al, 2008) of 8th grade students in public and private schools in the United States that compared the results of a mathematics assessment that included identical questions delivered either by web through a computer or with a paper-and-paper test found that paper-and-pencil delivery resulted in significantly higher scores. Greater computer familiarity by the participant resulted in higher online scores, and online scores showed greater variability. All demographic subgroups scored higher in the paper delivered test.

While the 2009 review (Macedo-Rouet, Ney, Charles, & Lallich-Boidin, 2009) was inconclusive on which delivery method was better overall, several trends were found in the literature. Studies with results that favored paper found more mental calculation errors with web delivery. Web results were stronger for high achievers and with open-ended questions but not for tests with multiple-choice questions. Tests with long reading passages were found to be more difficult on paper.

Computers for Content Delivery

Computer use in education is no way limited to the delivery and scoring of assessments. The effectiveness of its use for content delivery and homework practice has also been extensively studied. As with all things related to mathematics and digital devices, the research on the use of computers in the classroom, or as the classroom (online classes), has uncovered mixed results.

A 2009 quasi experimental study (Macedo-Rouet, Ney, Charles, & Lallich-Boidin, 2009) of 122 undergraduate students taking a course on mathematics tools for life sciences found that when comparing paper-based and web-based delivery of study materials, students' perception was that they performed better in using the paper delivery. Online practice quizzes did improve overall scores and web-delivery was perceived by the participants as more efficient than using printed materials, though the rate of knowledge acquisition was not different. Students expressed a preference for printed documents, and those who used printed documents produced the highest performance outcomes. Students' perception was that the paper materials required less effort to read. The researchers conjectured that web delivery might require a higher cognitive load. High achieving students benefited most from online delivery. Students had a positive view of the website.

While the study of undergraduate students found that web delivery provided a greater benefit to high achieving students, the use of an online mathematics application was found to have the greatest effect on low achieving 6th - 8th grade mathematics students (Chang, Evans, Kim, Norton, & Samur, 2015). In this quasi-experimental study, an online learning game delivered practice fractional problems to the treatment group. The control group performed paper-and-pencil drills of comparable items over the same period of time. Pretest-posttest results indicated that only the 7th grade group should significant gains over the paper-and-pencil group. Low achieving students benefited the most. The researchers concluded that the learning game was especially effective as a remediation tool.

A study conducting a similar comparison of mathematics practice (Hauk, Powers & Segalla, 2015) of 439 college algebra students found no significant difference in performance or achievement gains between students who used a web-delivered practice application and those who practice with paper and pencil. The software allowed the researchers to collect detailed information about student responses to questions, but was not designed to collect information on how students were thinking about the problems.

Computers for Conceptual Understanding

When digital technology is used as more than a delivery system for problem practice, it becomes possible to study the technology's effect on conceptual understanding. Several studies have been conducted with different populations to determine what effect enrichment with digital technologies has on conceptual understanding. The following research does not compare outcomes with paper-and-pencil delivery systems. The studies instead explore the extent to which digital technologies can enhance and enrich the understanding and the experience of the participants.

A qualitative study of 8th grade students (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012) found that computer tools strengthened students' understanding of functions. In this study students were not separated into computer and paper-and-pencil groups. The study integrated both types of tools within a program to determine if computer tools fostered the transition from structural to operational understanding of functions. While paper-and-pencil techniques were used within the course, computer integration was not used to replace physical tools, to but to support student understanding. The researchers found that the computer tools helped students integrate the operational and structural aspects of the concept of functions, supporting explorative and investigative activities.

Preservice and in-service teachers responses to the integration of digital technology into their coursework has also been studied. A multiple-case study of two prospective teachers explored their problem-solving strategies (Koyuncu, Akyuz, & Cakiroglu, 2015) when solving plane geometry problems. The study found that though the participants mostly used algebraic solutions when using paper-and-pencil, they preferred to solve problems using geometric solutions in the digital environment. Within the paper-and-pencil environment the participants demonstrated that they were more comfortable solving equations than exploring possible geometric relationships. In spite of the preference for using paper-and-pencil, participants were comfortable developing geometric relationships when within the digital environment. The researchers conjectured that the fact that geometric construction was slower and required more thought in the digital environment allowed participants to better absorb the nature of the problem. These results are similar to those that the researchers found with the 8th grade students who were exploring functions: technology allowed the participants to develop alternative strategies for problem solving and bolstered conceptual understanding.

Unlike the supportive symbiotic relationship between the physical and digital modes of problems solving found in the preceding studies, clinical interviews with four pre-service secondary mathematics teachers (Zembat, 2008) found that paper-and-pencil environments limited the participants modes of thinking when considering the concept of derivatives. The study found that misconceptions and limitations on their concepts of function, slope, and derivative were partly overcome when the participants worked within a digital environment. Additionally, when the participants used paper-and-pencil techniques their thinking was limited to analytical reasoning and did not shift toward creativity or practical reasoning. Most participants moved more freely between all three types of reasoning within the digital environment.

Online courses

Any comparison of the physical and digital worlds and their impact on mathematics learning would be incomplete without a discussion of online education. By 2010 35% of four-year higher-education departments and 88% of community colleges offered distance learning math courses (Trenholm, Alcock,& Robinson, 2016). Though online courses save institutional resources, and provide flexible learning experiences, institutions report low levels of student satisfaction and high rates of attrition (Xu & Jaggers, 2011). Though an exhaustive review of the research on online delivery is beyond the scope of this paper, it makes sense to address how digitally mediated interactions between students impacts students and instructors.

Changes in motivation in 100 high school students enrolled in an online self-paced mathematics course was measured using a survey administered to participants three times over the course of a semester (Kim, Park, Cozart & Lee, 2015). The purpose of the study was to determine what difference online delivery outcomes between high performing students and low performing students. As expected, high performing students maintained high levels of effort regulation and self-efficacy. The authors argued that self-efficacy can be improved through effort regulation. Perceptions of intrinsic value remained low in both groups, and both groups' uses of shallow strategies dominated by the end of the course. Students persisted even with low measures of enjoyment and interest.

A quantitative comparison of first year undergraduate mathematics students' achievement (Fonalohi, Kahn & Jokhan, 2014) found that there was no significant statistical difference in the pass rate between the students who were studying in the face-to-face classes and those who were taking the course online. A close look at the breakdown of student grading revealed that students studying online achieved higher marks on coursework, but lower scores on exams compared those in the face-to-face classes. The researchers concluded that the students were receiving equivalent learning in different ways. A second quantitative study of college engineering students (Mativo, Hill & Godfrey, 2013) found that delivery was format independent and that there was no significant difference in the success of students who received instruction face-to-face and students who received instruction online.

The perspectives of undergraduate mathematics instructors of distance-learning courses were explored in a survey of 70 undergraduate mathematics instructors and semistructured interviews of six instructions (Trenholm, Alcock,& Robinson, 2016) The researchers found that instructors were concerned about loss of face-to-face human interaction. All participants had extensive face-to-face experience and were teaching at least one introductory undergraduate mathematics course. Participants believed that their face-to-face practices were effective and felt that it was important to replicate or replace them in online courses. They expressed concern about the loss of interactions with student.

IPads

The iPad has been the most quickly adopted digital technology in history. By 2011 more than 1.5 million iPads were being used specifically for education with access to more than 20,000 education applications specifically for the device. It was assumed that the technology had the potential to fundamentally change learning and teaching (Alyahya & Gail, 2012). Because the introduction of the iPad to the education is still relatively recent, there is a limited amount of research available on iPads and other touchscreen devices..

A quantitative study of prekindergarten four and five year old children (Bates, Shifflet, Latham, Ennis & Matton, 2015) compared outcomes after instruction with either traditional physical, or digital manipulatives. Participants were randomly assigned into either the traditional manipulatives group or the digital manipulatives group. Both groups were taught by the same teacher who made the instruction as similar as possible. After six weeks of instruction the groups were assessed. Though both groups showed significant improvement in their abilities, no significant difference was found between the computational scores after receiving instruction. The researchers concluded that both digital and traditional manipulatives were effective tools to build children's computational skills.

Eighty first-grade students were the participants in a quasi-experimental control design study of the effect of iPads on mathematics achievement (Al-Mashaqbeh, 2016). The participants were divided into two groups. The control group received traditional mathematics teaching, and the experimental group used iPads to study and practice math topics. The pretest-posttest design allowed the researcher to determine that there were significant differences between the experimental group and the control group with the experimental group scoring significantly higher than the control. The researcher determined that the higher scores were the result of the greater access to materials and activities that the iPad allowed the participants.

Game based applications were used in quasi-experimental study of fifth-grade students (Carr, 2012). As with the study of the first-grade students, a pretest-posttest design was used to compare the results of a nine-week study that provided iPads with mathematics games to the treatmentment group, and traditional mathematics course delivery to the control group. The interaction with the iPads did not significantly affect student achievement: There was no significant difference between the treatment group and the control group.

At the high school level, an exploratory case study study of five students used iPads as a way to manipulate geometric objects (Arzarello, Bairral & Dane, 2014). The tablets provided rapid interactivity through motion feedback. Participants demonstrated the ability to both construct and manipulate objects on the screen to solve problems. The researchers observed that when students manipulated objects they were focused on conceptual understanding and reasoning. They determined that the touch screens allowed for interaction and manipulation that could not be achieved with a computer mouse, leaving room for further study of the students geometric conceptualizations in the tablet environment.

Embodied cognition

Husserl (2005) described pure possibility as imagined objects that exist in such a way that we experience them with the same power and sense of reality as we experience concrete objects: that recall of the objects are as valid in memory as experiences in the physical world. He conjectured that new possibilities can be fashioned from the imagined object through manipulation, allowing for limitless possibilities. Husserl's description of how we perceive of abstract ideas was later supported by a seminal neuroscience study (Gallese & Lakoff, 2005) that found evidence that conceptual knowledge is mapped within our sensory-motor system, and that this characterizes the way that we function in the world. The same structure that moves us and gives us structural perceptions also structures abstract thought. This form of embodied perception is described as embodied cognition.

Nemirovsky and Ferrara (2009) integrate Husserl's ideas of pure possibilities in their description of mathematical imagination. Unlike the complete conceptual freedom that pure possibility allows, mathematical imagination is bound by the logical necessity inherent in mathematics. The use of symbols to prove mathematical entities integrates pure possibilities with the empirical or physical world. Their study of 21 10th and 11th grade algebra students used body motion and technology to integrate movement and mathematics learning, exploring the possibility that mathematical rules and propositions can reside in the body. The resulting student discussions demonstrated multiple points of view and explorations of mathematics that incorporated both space and time.

Hutto, Kirchhoff, and Abrahamson (2015) also used the concept of mathematics as embodied practice in a review of a prior experimental research study (Reinholz, Trninic, Howison & Abrahamson, 2010). The research study compared the conceptual understanding of participants who had be taught to move their arms and hands in a way that would enact proportional relationships. The treatment group significantly outperformed the control group on measures of conceptual understanding of proportional reasoning. With the emergence of touch technology the authors call for renewed attention to the real and practical implications of embodied cognition in the domain of mastery of mathematics.

The question of whether embodied cognition extends to the affordance of gesture control that tablets provide was explored in the a study of 61 8-11 year olds (Agostinho, Tindall-Ford, Ginns, Howard, Leahy, & Paas, 2015). The control group was instructed to look at information that was highlighted and circled on an iPad app. The treatment group was instructed to trace information that was highlighted and circled on an iPad app using their index finger. Participants who traced the graphs with their fingers achieved a higher performance result on a test of recall than participants who studied the material without tracing. The authors conclude that finger tracing can support problem-solving on a tablet device and that gesturing has a fundamental impact on a range of educationally relevant cognitive functions.

Touchscreen tablets allow for a digital version of sensorimotor interaction to be integrated into mathematics learning activities. In the digital environment, much as Husserl described pure possibilities, mathematical symbols evolve from things with which you act to things with which you think, becoming frames of reference for mathematical reasoning. Case studies of tutor-student behaviors in embodied-interaction learning environment (Abrahamson & Sanchez-Garcia, 2016) found that students who develop sensorimotor schemes, as a real or imagined objects or aspects, can shift them into explicit mathematical re-visualization of the environment.

Technology is changing at an astonishing rate and there is a scarcity of research on interactive mathematics learning. Consequently the integration of digital technological advancements with educational institutions is not keeping pace, causing a misalignment between theory and changing practices. A dynamical systems approach to learning is emerging which views learning processes as goal oriented adaptive interactions in the environment (Abrahamson & Sanchez-Garcia, 2016) rather than disembodied symbolical propositions. This motor-action theory describes learning as emergent, systemic, nonlinear, distributed, and self-adaptive; varying within and across individuals. Learning occurs when cognitive structures emerge from recurrent sensorimotor patterns, repeated patterns of movement, enabling action to be perceptually guided. Mathematical meanings emerge from guided, situated motor-action coordinations. Abrahamson and Sanchez-Garcia describe embodied-interaction as similar to coaching physical disciplines.

Discussion

It has been demonstrated that tangible interaction with relational objects produces higher performance in the acquisition of conceptual understanding of mathematics (Goodman, Seymour & Anderson, 2016). Software and hardware advances in technology have reached the point where digital delivery methods can mimic forms and experiences that exist in the physical world making the distinction between forms more meaningless with the passage of time. While writing on a touchscreen still does not match richness of the physical sensation of moving a pencil over paper, the advantages of paper-and-pencil over digital technology continue to diminish.

What are the affordances of paper-and-pencil that are so compelling? A study (Zuckerman & Gal-Oz, 2013) found that users preferred a tangible interface for a digital device that was physically manipulated to a graphical interface that was manipulated with a mouse. Both versions resulted in equal in performance quality and task completion. Users stated that they preferred the tangible interface and found it more enjoyable because it allowed for physical interaction, rich feedback and realism. The theory of embodied cognition may give us ways to understand the advantages of moving a writing instrument over a surface that provides haptic feedback. It has been proposed that if a digital version of an object can be developed that mimics critical aspects inherent in a physical object, the advantage of the physical object is mitigated (Sheu and Chen, 2014).

Since the beginning of this century we have moved from an environment where digital devices were being introduced into classrooms and instructors were being challenged to integrate their use (Cuban, 2001), to the ubiquitous presence of digital devices (Goodman, Seymour & Anderson, 2016). Costs and benefits remain understudied. iPad adoption was recently seen as a progressive move for an institution. But what was seen as groundbreaking may turn out to be a footnote in the history of education. At the pace that things are changing we may not have time to study the impact of iPads on the education before they disappear. The march of technology has become a sprint, and what was progressive yesterday borders on the obsolete today.

Though than one third of United States middle and high school students use school issued mobile devices, and 60-70% do school work on some type of mobile device (Goodman, Seymour & Anderson, 2016; Grant, 2015), the nature of the device may be about to change. A recent decline in Apple's sales of iPads (Bailey, 2016) may indicate not only market saturation, but may also signal another trend. Small mobile screens are quickly replacing their larger precursors. The availability of broadband internet access in students' homes is becoming less likely as access to the internet becomes increasingly mobile (Fung, 2015). The Pew Research center (Horrigan & Duggan, 2015) found that broadband at home has plateaued and that more users rely on smartphones for online access. Most significantly African Americans, low household incomes, and those living in rural areas increasingly rely on mobile devices for internet access. Cost is the major reason. Though many users see the lack of broadband access as a disadvantage, more users have access to the internet only through their smartphones. A July 2015 survey results showed that 68% of United States residents have a smartphone. Of those,13% rely on smartphone at home for internet access. These users tend to be younger, lower-income, and non-white. Cost is a substantial challenge, and users state that monthly fees keep them from acquiring broadband access at home.

In spite of the high rates of adoption, digital literacy remains a problem (Chen, Seilhamer, Bennett & Bauer, 2015). Only 30 percent of instructors incorporate mobile technologies into assignments. And 55% ban or discourage their use in the classroom, perceiving small screens as a distraction. A survey of university students found that over 95% owned a smartphone. Tablet ownership was much lower at 57%. Demographics did not affect ownership use. Younger students reported greater use of smartphones. Social networking, music, social media were the top three uses reported in the survey. Education was 14th with only 19% of users reporting educational apps as their most frequently used apps. 73% reported using smartphones for learning on their own. 66% used a mobile app at least once each week. 35% of users said they did not want instructors to integrate mobile applications into courses because of technical support issues.

The Horizon Report is an "ongoing research project designed to identify and describe emerging technologies likely to have an impact on learning, teaching, and creative inquiry in education." The 2014 report (Johnson, Adams, Becker, Estrada, & Freeman, 2014) identified the importance of social media on communities of practices and predicted that gamification would gain support among educators as a training and motivation tool. By 2016 (Johnson, Adams, Becker, Estrada, Freeman & Hall, 2016) their focus had also shifted to mobile technologies and a prediction that students and instructors would make greater use of the devices, with a warning for the need for greater pedagogical support to understand how to use mobile technology for learning purposes.

In spite of continuing advances of technology, the arguments over whether the presentation of mathematics learning environments are better situated within the physical or digital worlds will persist. Resistance to change is part of human nature. More than this, the pleasures of interactions that stimulate our senses and ground us in place and time are still overwhelmingly present in the physical world that surrounds us when all of the devices have been turned off. Mathematics is a technology that humans developed to allow us to describe the physical world in exquisitely finite and accurate ways. Though I have touched on the idea of mathematics as an object in this paper, mathematics is most often perceived as a tool.

At its most basic, digital technology too is only a tool. Computers, tablets, and smartphones all deliver content that mimics the physical world. Beyond the delivery of content these devices allow for communication in ways that could hardly have been imagined a few years ago. Ultimately how a tool is deployed is more important than the tool itself. Hardware is mute and useless without software. A discussion of software has been beyond the scope of this paper, but not beyond the scope of the argument. The puzzle of how best to leverage digital technologies for education has yet to be solved.

Most likely as new ways of distributing and acquiring knowledge present themselves the ways that have served us well, and continue to serve us well, will not be supplanted. Paper-and-pencil processes will remain with us, supported by the newest and best technologies.

References

Abrahamson, D., Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences*, *25*(2), 203-239. doi: 10.1080/10508406.2016.1143370

Agostinho, S. S., Tindall-Ford, S., Ginns, P., Howard, S., Leahy, W., & Paas, F. (2015). Giving learning a helping hand: Finger tracing of temperature graphs on an iPad. *Educational Psychology Review, 27*(3), 427-443.

Al-Mashaqbeh, I. F. (2016). IPad in elementary school math learning setting. *International Journal Of Emerging Technologies In Learning*, *11*(2), 48-52. doi:10.3991/ijet.v11i2.5053

Alyahya, S., & Gall, J. E. (2012). iPads in education: A qualitative study of students' attitudes and experiences. *World Conference on Educational Multimedia, Hypermedia and Telecommunications*, 1266-1271.

Arzarello, F., Bairral, M. A., & Danè, C. (2014). Moving from dragging to touchscreen: Geometrical learning with geometric dynamic software. *Teaching Mathematics & Its Applications*, *33*(1), 39-51.

Bailey, B. (2016, April 26). Apple reports iPhone sales down, 1st revenue drop since 2003. *The Washington Post.* Retrieved from https://www.washingtonpost.com/national/apple-reports-iphone-sales-down-1st-revenue-drop-since-2003/2016/04/26/eb252b0e-0bef-11e6-bc53-db634ca94a2a_story.html

Bates, A., Shifflet, R., Latham, N., Ennis, S., & Mattoon, C. (2015). Examining computational skills in prekindergarteners: The effects of traditional and digital manipulatives in a prekindergarten classroom. *Early Childhood Research & Practice (ECRP)*, *17*(1), 4.

Bennett, R.E., Braswell, J., Oranje, A., Sandene, B., Kaplan, B., & Yan, F. (2008). Does it matter if I take my mathematics test on computer? *A Second Empirical Study of Mode Effects in NAEP. Journal of Technology, Learning, and Assessment, 6*(9). Retrieved from http://www.jtla.org

Carr, J. M. (2012). Does math achievement h'APP'en when iPads and game-based learning are incorporated into fifth-grade mathematics instruction?. *Journal Of Information Technology Education*, *11,* 269-286.

Chang, M., Evans, M. A., Kim, S., Norton, A., & Samur, Y. (2015). Differential effects of learning games on mathematics proficiency. *Educational Media International*, *52*(1), 47-57.

Chen, B., Seilhamer, R., Bennett, L., & Bauer, S. (2015). Students' mobile learning practice in higher education: A multi-year study. *Educause Review*. Retrieved from http://er.educause.edu/articles/2015/6/students-mobile-learning-practices-in-higher-education-a-multiyear-study

Clemson (2016). Math 1060 Calculus of 1 Variable. Retrieved from https://mthsc.clemson.edu/ug_course_pages/view_course_page.py?course_id=6

Cooke, R. L. (2011). *The history of Mathematics: A brief course*. New York: John Wiley & Sons, Inc.

Cuban, L. (2001). *Oversold and Underused: Computers in the Classroom.* Cambridge, MA: Harvard University Press

Doorman, M. M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: from repeated calculations to functional thinking. *International Journal of Science & Mathematics Education*, *10*(6), 1243-1267.

Dunham, P. H., & Dick, T. P. (1994). Research on graphing calculators. *The Mathematics Teacher, 87*(6), 440-445.

Drijvers, P., & Doorman, M. (1996), The graphics calculator in mathematics education, *Journal of Mathematical Behavior, 15*(4), 425-440.

Fung, B. (2015, April 18). New data: Americans are abandoning wired home internet. *The Washington Post*. Retrieved from https://www.washingtonpost.com/news/the-switch/wp/2016/04/18/new-data-americans-are-abandoning-wired-home-internet/

Gallese, V., & Lakoff, G. (2005). The brains's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*(3–4), 455–479. doi:10.1080/02643290442000310.

Goodman, S. S., Seymour, T. L., & Anderson, B. R. (2016). Achieving the performance benefits of hands-on experience when using digital devices: A representational approach. *Computers In Human Behavior*, *59,* 58-66.

Hauk, S., Powers, R. A., & Segalla, A. (2015). A comparison of web-based and paper-and-pencil homework on student performance in college algebra. *Primus: Problems, Resources & Issues In Mathematics Undergraduate Studies*, *25*(1), 61-79. doi:10.1080/10511970.2014.906006

Hargreaves, M., Shorrocks-Taylor, D., & Swinnerton, B. (2004). Computer or paper? That is the question: does the medium in which assessment questions are presented affect children's performance in mathematics? *Educational Research*, *46*(1), 29-42.

Hitt, F. (2011). Construction of mathematical knowledge using graphic calculators (CAS) in the mathematics classroom. *International Journal Of Mathematical Education In Science & Technology*, *42*(6), 723-735.

Horrigan, J. B., & Duggan, M. (2015). Home broadband 2015. December 21, 2015. Retrieved from http://www.pewinternet.org/2015/12/22/2015/Home-Broadband-2015/

Husserl, E.(2005) Pure possibility and phantasy. *Phantasy, Image Consciousness, and Memory Volume 11 of the series Edmund Husserl Collected Works* pp. 659-688.Netherlands: Springer doi: 10.1007/1-4020-2642-0_19

Hutto, D. D., Kirchhoff, M. k., & Abrahamson, D. D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. *Educational Psychology Review*, *27*(3), 371-389.

Johnson, L., Adams Becker, S., Estrada, V., & Freeman, A. (2014). *NMC Horizon report: 2014 higher education edition*. Austin, Texas: The New Media Consortium

Johnson, L., Adams Becker, S., Cummins, M., Estrada, V., Freeman, A., and Hall, C. (2016). *NMC Horizon Report: 2016 Higher Education Edition.* Austin, Texas: The New Media Consortium.

Kim, C. M., Park, S. W., Cozart, J., & Lee, H. (2015). From motivation to engagement: The role of effort regulation of virtual high school students in mathematics courses. *Journal Of Educational Technology & Society*, *18*(4), 261-272.

Kop, P. M. G. M, Janssen, F. J. J. M., Drjvers, P. H. M., Veenman, M. V. J. & van Driel, J. H. (2015). Identifying a framework for graphing formulas from expert strategies. *The Journal of Mathematical Behavior, 39*, 121.134. doi:10.1016/j.jmathb.2015.06.002

Koyuncu, I., Akyuz, D., & Cakiroglu, E. (2015) Investigating plane geometry problem-solving strategies of prospective mathematics teachers in technology and paper-and-pencil environments. *International Journal of Science And Mathematics Education, 13*(4), 837-862.

Macedo-Rouet, M., Ney, M., Charles, S., & Lallich-Boidin, G. (2009). Students' performance and satisfaction with Web vs. paper-based practice quizzes and lecture notes. *Computers & Education*, *53*(2), 375-384.

Morris, W. (2014). Syllabus. Retrieved from http://math.gmu.edu/~wmorris/syll113s14.pdf

Mativo, J. M., Hill, R. B., & Godfrey, P. W. (2013). Effects of human factors in engineering and design for teaching mathematics: A comparison study of online and face-to-face at a technical college. *Journal Of STEM Education: Innovations & Research*, *14*(4), 36-44.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM

National Council of Teachers of Mathematics (2014). *Principles to Action.* Reston, VA: NCTM

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Retrieved from http://www.corestandards.org.

Nemirovsky, R., & Ferrera, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70,* 159-174. doi10.1007/s10649-008-9150-4

Petroski, H. (1990). *The Pencil: A history of design and circumstance*. New York: Alfred A. Knopf, Inc.

Reinholz, D., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: embodied artifacts and the guided emergence of mathematical meaning. In P. Brosnan & D. Erchick (Eds.), *Proceedings of the Thirty-Second Annual Meeting of The North-American Chapter of The International Group For The Psychology Of Mathematics Education 32,* 1488–1496.

Sheu, F.-R., & Chen, N.-S. (2014). Taking a signal: a review of gesture-based computing research in education. *Computers in Education, 78*, 268–277.

Steen, L. 1. (1987). Who still does math with paper and pencil?. *Chronicle of Higher Education*, *34*A48.

Syllabus (2016). MATH 140, Calculus with Analytic Geometry I. Retrieved from https://www.math.psu.edu/files/140syllabusSP16.pdf

Trenholm, S., Alcock, L, & Robinson, C. (2016). The instructor experience of fully online tertiary mathematics: A challenge and an opportunity. *Journal for Research in Mathematics Education, 47*(2), 147-161.

Usiskin, Z. (1998). Paper-and-pencil algorithms in a calculator-and-computer age. *Yearbook (National Council Of Teachers Of Mathematics)*, *1998*7-20.

Xu, D., & Jaggars, S. S. (2011). The effectiveness of distance education across Virginia's community colleges: Evidence from introductory college-level math and English courses. *Educational Evaluation and Policy Analysis, 33*(3), 360 –377. doi:10.3102/0162373711413814

Zembat, I. O. (2008). Pre-service teachers' use of different types of mathematical reasoning in paper-and-pencil versus technology-supported environments. *International Journal Of Mathematical Education In Science & Technology*, *39*(2), 143-160.

Zuckerman, O., Gal-Oz, A. (2013) To TUI or not to TUI: Evalating performance and preference in tangible vs. graphical user interfaces. *International Journal of Human-Computer Studies 71*(7), 803-820