Teaching and Learning Trigonometry:

A Review of the Literature

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Trigonometry, as its name describes, is the mathematics of the measure of triangles. On the trajectory of formal mathematics in high school, trigonometry is taught in high school after algebra and as a precursor to calculus. Ideally, a working knowledge of trigonometry should provide the skills and conceptual understanding that allow a student to extend secondary-level geometry and algebra fundamentals into the prerequisite mathematics necessary for the study of college level physics, architecture, engineering, and computer science (Byers, 2010; Harris, 1940; Weber, 2005). Trigonometry is more than a stepping-stone to further mathematics. The understanding of trigonometric function provides the framework that makes the experienced universe more understandable (Raimi, n.d.). The mathematics of trigonometry allows us to measure the height of mountains, to track the movement of the universe, and to measure the waveforms of light and sound.

There are two contexts in which trigonometric functions are presented that students must master in order to successfully navigate the subject: the right triangular, or ratio form; and the unit circle form (Demir & Heck, 2013; Moore, 2012). The trigonometric functional values of the right triangular form are designated by the degree measure of the acute angles of a right triangle and are calculated using the ratios of the sides of the triangle. Current applications of right angle trigonometry include surveying, navigation, and triangulation. Trigonometric functional values are represented in the circular context by the relationship of the section of arc of the circle that is subtended, or fits within, pairs of radius lines drawn from the center of the circle to the associated line segment. Periodic, or regularly repeating, trigonometric functional values come from consecutive circular movement of the radius line segment around the circle. Current

Courtney Baker: A nice transition into the next section. Helps with the flow of the paper.

applications of circular, or periodic trigonometry include the study of waveforms in physics and engineering and periodic functions used in social science (Bressoud, 2010). Though the triangular form is currently taught before the circular form, the history of the development of the circular form predates the triangular form.

History

The earliest known practical application of the circular form of trigonometric functions was Hipparchus' use of triangle ratios in the 2nd century BCE to explain the unequal lengths of the seasons. Hipparchus conceived of triangles formed within the circuit of the sun using the location of three points: one at the center of the circle, and two that subtend the arc traveled by the sun in during course of each season. Using the proportions of the sides of the triangles formed in this manner he was able to accurately calculate the seasonal relative distances from the earth to the sun (Bressoud, 2010). Drawing a line between the two points that subtend an arc forms a cord. By the end of the 2nd century CE the proportion of arc lengths to the chord had been refined by Euclid and Ptolemy into a table of approximate values in half-degree increments of the full 360-degree rotation (Boyer, 1991; Bressoud, 2010). In the 18th century Euler fixed the magnitude of the radius at a unit of one and used the unit of measure for the arc, resulting in a measured distance of the arc of a complete circle of 2π radaii. It was another 100 years before functions shifted from the measure arc length in number radii, to the measure of the angle in radians (Bressoud, 2010), a shift from a measure of distance to a measure of magnitude. This allowed the circular form to be applied to periodic functions.

Ptolemy used the trigonometry of triangles in the second century CE to calculate the angle of the sun by calculating the ratio of height of an object to the length of its shadow. The first known table of triangle values that related the ratios of lengths of sides to the measure of angles was not developed until the 8th century by Al-Khwarizimi of Baghdad (Van Brummelen, 2009). The publication of a Latin translation of a Greek text that details the application of trigonometry to the calculation of the sides of right triangles in the16th century marks the beginning of the common use of the trigonometry of triangles in surveying. It was Euler's technique of fixing the radius at a unit of one in the unit circle that allowed for the development of our current understanding of triangular trigonometric functions as the ratios of the sides of any right triangle (Bressoud, 2010; Van Sickle, 2012).

In the United States before the late nineteenth century, trigonometry was primarily taught as the relationship between arcs and angles: as an extension of circular trigonometry. As evidenced in textbooks from the time (Van Sickle, 2012) the teaching of trigonometry shifted from the circular form to the triangular form as a way to make the subject more accessible to students. The earliest of the textbooks that demonstrate this shift include proofs that connect the development of the ratio method to the relationship between arcs and angles and discuss the periodic nature of trigonometric functions[redacted]the turn of the century the subject had in some cases been reduced to only the relationship between lines and angles present in the right triangular context.

Courtney Baker: Think about the implications for teaching and learning that minimizing the richness of mathematics has – even at the turn of the century!

The right triangle and unit circle forms of trigonometric functions are taught at different times and as unrelated concepts in the sequence of formal mathematics in the United States. The triangular form is generally taught in high school geometry classes (Blackett & Tall, 1991), and is part of the designated standards for geometry in the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The Standards do not designate a subject matter context for the circular form of trigonometry, placing the form within the domain of high school functions. The unit form is usually part of the curriculum of high school or postsecondary precalculus classes (Thompson, 2008). The recommended sequences presented in Appendix A of the Common Core State Standards place the triangular form trigonometric functions in geometry, and the circular form of trigonometric functions in either algebra II or precalculus (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Courtney Baker: The second time you use a citation like this you can abbreviate.

Content

Right triangles

Trigonometric functions were simplified to the ratios of the sides of a right triangle to make the materials more accessible to students more than a hundred years ago. The perception that the triangular form is the more basic and therefore simpler form (Bressoud, 2010; Van Sickle, 2012), appears to be the reason that trigonometry is currently taught in the triangle-circle sequence.

On the surface, the decision to place the triangular form of trigonometric functions within the geometry curriculum seems a logical one. Geometry is by definition the study of the properties of angles, triangles, circles, and other geometric forms. But, the functions of the ratios of right angle trigonometry are not directly related to the properties studied in geometry (Moore, 2012), which may explain the common use of the mnemonic SOH-CAH-TOA to represent the trigonometric ratio values of the acute angles of a right triangle. The device is demonstrably useful, allowing students to correctly solve problems in a context that has no grounding in their previous mathematic training. Trigonometry is the first functional form that students will encounter that cannot be solved with familiar algebraic formula calculations (Bressoud, 2010). There is nothing in the literature about the development of the common use of SOH-CAH-TOA

Courtney Baker: What does this stand for? Explain any acronyms.

Courtney Baker: Or was it easier to teach? Think about the impact of losing the connections at the turn of the century?

Courtney Baker: Absolutely!

to represent trigonometric ratio values. Perhaps it was developed as a way to ease students' transiti[redacted] to the use of the unfamiliar functional form.

The mnemonic assists students in finding the function of *sine*, *cosine*, and *tangent* of an angle of a right triangle, defined by the ratios of the sides: *sine* as the ratio of the measure of the opposite leg to the measure of the hypotenuse (SOH), *cosine* as the ratio of the measure of the adjacent leg to the measure of the hypotenuse (COS), and *tangent* as the ratio of the measure of opposite leg to the measure of the adjacent leg (TOA) (Thompson, Carlson, & Silverman, 2007). While this method allows students to accurately calculate the side lengths and angle measures of right triangles, there is no evidence that students associate calculated results with the functional relationship between the measure of the angle and the ratio of the sides (Bressoud, 2010). Weber's 2008 review of the existing research on the teaching and learning of trigonometry found that the challenge of relating functional relationships without performing algebraic operations was a challenge for students. The satisfactory results of the procedural application of the mnemonic mitigates students' need for a conceptual understanding of the relationship between the measure of an angle and the resulting trigonometric values in order to solve the problems they are presented.

The conceptual limitations of the SOH-CAH-TOA model does not only affect students' understanding, there is evidence that the model also has a negative impact on the conceptual understanding of in-service an[redacted]-service teachers. A study of 14 secondary mathematics teachers of trigonometry (Thompson, Carlson, & Silverman, 2007) found that for many, their conceptual understanding of trigonometry was limited to the SOH-CAH-TOA model. Not only were the teachers unable to assign values to angles other than those present in right triangles, there was no indication that the teachers understood the functional relationship between the measure of the angle and the values present in the right triangle environment. A 2014 (Yigit) study of five pre-service math teachers found that their understanding of trigonometry was limited to memorized ratios and reliance on mnemonics to solve right triangles. There was no indication of an understanding of the relationship between angle measures and the proportions of the side lengths of a right triangle. Incorrect memorization resulted in incorrect results. There was evidence that unconventional memorization regarding triangle ratios may have prevented the participants from constructing schema level understanding by masking the relationship between angles and sides in a right triangle.

One study (Kendal & Stacey, 1996) is often cited as support for the introducing trigonometric functions using the ratios of right triangle before presenting unit circle trigonometry. The study was a causal-comparative study of eighty-eight tenth grade students that compared the skills of students who were taught the right-triangle method with students who began their study of trigonometric functions with the circular form. The students who were taught trigonometric functions in the right triangle context significantly outperformed the student whose initial training was in the circular context on tests of their ability to solve triangles by calculating their angle measures and side lengths. Conceptual development was not tested. While these results have been cited (Weber, 2008) as a reason to support the triangle-circular sequence, the fact that the content of the pretest and posttest was limited to the task of solving triangles by calculating their angle measures and side lengths limits the applicability of the results to that very narrow context. The results do not provide evidence that the triangular-circular sequence is preferable in the more complex context of circular periodic trigonometry. Similarly, the end-of-year exam in which the students that were taught the triangular form scored significantly higher than those who had been taught the circular form also only tested students' ability to solve

Courtney Baker: This demonstrates your ability to deeply analyze the literature.

triangles. At the time of the study the standard method of using trigonometric functions to solve triangles was to place the triangle within the first quadrant of the Cartesian coordinate plane, using the trigonometry of circles limited to the first quadrant. The researchers argued that this presentation was more complex than necessary and provided no long-term benefit to students. They recommended a change to a curriculum that used the simplified right angle representation. While there is evidence to support this claim if the measure of student achievement is limited to the measure of right triangle side lengths and angles, lack of further research has given us no ability to determine the long-term effects. There is no indication of whether or not these students moved into mathematics courses that presented the trigonometric functions in the complete circular form, and therefore no evidence of how either group may have performed in the circular-periodic context.

Though the Common Core State Standards for geometry (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) do not include knowledge of the ratios of the sides of 30-60-90 and 45-45-90 triangles, designated special triangles, the ratios of the special triangles are often taught in tandem with the right triangle form of trigonometric functions. The ratios of these special triangles are memorized and used as additional tools for students to solve triangles. Beyond the identified number of degrees in the acute angles of the right triangle, expanded ideas of angle measure are not explored (Thompson, 2008).

The Unit Circle

The trigonometry of the unit circle is taught to mathematic students after they learn triangle trigonometry, usually in a subsequent course in a later year. The unit circle is created by placing the center a circle with a radius length of one unit on the origin of a Cartesian coordinate plane. The circle has two sets of terminal points located on the circumference of the circle. One set of terminal points is placed at locations that divide the circumference of the circle into eight equal arcs; the other set of terminal points is placed at locations that divide the circumference of the circle into 12 equal arcs. The points coincide where the circle intercepts the x and y axes. Radii lines drawn from the origin to the terminal points form rays that subtend the arc that defines the trigonometric function. The measure of the arc can be given either in degrees or radians which use radius as a unit of measure. A complete rotation is comprised then of either 360 degrees or 2π radians (Moore, 2012). The measure of the arc subtended by the rays determines the functional value. With a radius value of one unit *sine* and *cosine* values become measures of magnitude rather than measures of length. The *sine* value is given by the vertical displacement, or y value; and the *cosine* is given by the horizontal displacement, or x value; producing the coordinate pairs associated with the unit circle. (Moore, 2012; Thompson, 2008).

The periodic nature of trigonometric functions in a circular context is defined by the angles that are formed by the circular sweep of the radius. The measure of the angle has none of the limitations of the right triangular form and can extend beyond the 2π arc of the complete circle to infinity in both the positive counter clockwise direction and the negative clockwise direction. The use of radians to designate the amount of openness of the angle allows the unit circle to show both magnitude and positive and negative direction as a function of the angle (Demir & Heck, 2013). Properties of similar triangles learned in geometry can be used with trigonometric functions in the unit circle context to examine the effect of multiple measures of amplitude by multiplying of the length of the radius and noting the resulting changes to *sine* and *cosine* values (Moore, 2012). The functional nature of trigonometry is revealed when periodic function graphs, such as sine waves, are created by graphing the coordinate values of the terminal points as functions of the angle of rotation (Thompson, 2008).

The difference between the familiar number schema of quantity measures and the measures of magnitude of in the unit circle can be a source of confusion for teachers and students (Moore, 2012). The inability to comprehend of the meaning of magnitude and its use quantitative analysis can inhibit students' ability to understand the meaning of operations on numbers (Reed, 2006). Training in the concept during introduction of the unit circle has been shown to effectively increase understanding (Moore, 2012). A case study of two secondary pre-service teachers (Moore, LaForest, & Kim, 2012) found that the teachers were unable to demonstrate the ability to reason about the magnitude of a unit. Consequently they were unable to demonstrate an understanding of the relationship of the unit circle and the measure of the radius resulting in the inability to provide quantitative meanings to their calculations. Without an understanding of magnitude they were not able to use the unit circle as a tool for reasoning. In a discussion of his prior five years of research on teacher and student understanding of trigonometric functions, Moore (2012) describes the importance of the understanding of unit magnitude of a unit in order to make measurement changes in amplitude when the values of trig functions displayed on the circle are graphed in wave form. Moore found that with practice, pre-service teachers were able to demonstrate an understanding of unit magnitude.

Even in the dynamic environment of the unit circle, rote memorization is often employed by students and teachers to construct the circle with the associated coordinate values. Though the relationship between the measure of the angle and the sine and cosine functions are explicit in the unit circle in ways that cannot be expressed in the triangle form of trigonometric functions (Moore, 2012), memorized constructs of the unit circle were found to prevent the full development of a schema of trig ratios in pre-service teachers (Yigit, 2014). Yigit (2014) found that in the circle form, as in the triangle form, error in memorization resulted in errors in results.

Integration

Right angle trigonometry is firmly established as the first form of trigonometry that students in the United States will encounter. There is no evidence of a perceived need to change the sequence. Moore (2012) found that issues of coherence often arose resulting in a disconnected understanding of trigonometry as students made the transition from right triangle trigonometry to circular trigonometry. This is to be expected in the segmented discontinuous way that students are currently being taught (Thompson, Carlson, & Silverman, 2007; Weber, 2005). Current curricular approaches do not address the lack of coherence, connection, and meaning between the two contexts that results in a shallow understanding of concepts (Demir & Heck, 2013; Moore, 2012; Weber, 2005, 2008). If students are to build a meaningful construct of trigonometric functions, the first experience of the triangular form of trigonometry must inform and strengthen student understanding of the circular form of the functions. Contextual interrelationships between the forms must be developed if this is to occur.

Within the existing environment there is ample opportunity to create a system of meanings that will allow reasoning about trigonometric functions in the multiple contexts in which they are found. Students' difficulty in acquiring an integrated understanding of trigonometry begins with the confusion surrounding the meaning and the measure of angles. Angles are measured in degrees in right angle trigonometry, and in radians in the unit circle in US textbooks (Moore, 2012). Confusion in the understanding of the measure of angles is not limited to high school students. Moore (2009) found that the understanding of angle measure in undergraduate pre-calculus students was related to perpendicular lines, geometric objects, and comparisons with supplementary and complementary angles. They were unable to solve tasks that required reasoning about angle measures independent of other angle measures. This level of understanding would be adequate and unchallenged by right triangle trigonometry, but would not be adequate for an understanding of the unit circle form.

A more sophisticated, universally applicable concept of the measure of an angle is necessary to provide the foundation needed to understand trigonometry in all of its contexts. The measure of an angle as the measure of its openness, rotation, or the arc that it subtends is the conceptual form that can be applied both contexts. If the measure of an angle is only procedurally calculated in degrees or radians, the concept of openness necessary for an adaptable understanding cannot be conceptualized. The introduction to angle measure is taught early as a fraction of 360 degrees, often with a ruler and a protractor. The protractors' shape and form embodies the measure of angle as the length of an arc, though it is not presented and used in this manner. Conceptualizing the measure of an angle as the measure of the arc that the angle subtends would allow students to view angle measure as a measure of openness and provide a foundation for coherence between right angle and circular trigonometry (Moore, 2012; Thompson, 2008). The understanding of angle as a rotation as a both a fraction of 360 degrees and radian measure would attain coherence if the length of the subtended arc is measured in both degrees and radian, allowing the relationship between the acute angles of the triangular form to the unrestricted angle measures of the unit circle to become evident (Bressoud, 2010; Thompson 2008). A case study (Moore, 2014) of a precalculus undergraduate student found that a student who initially did not have a unified conception of angle measurement came to understand the length of a subtended arc as a fraction of a circle's circumference in both degrees and radians.

Courtney Baker: This section really does the literature justice. It details the study just enough – while also connecting to the broader theme of the section.

The transition between studies is also very clear, and not choppy at all.

Consequently, he was able to think of radians as units of magnitude, successfully calculating answers in contextualized "real world" problems.

In an effort to address the arc problem, a study of 24 pre-university students (Demir and Heck, 2013) were presented a metaphor for arc length as the distance traveled around the rim of a geometric object. The purpose of the study was to develop a means to integrate the multiple forms of trigonometry. Though viewing arc length as a journey and distance traveled allowed students to develop a deep understanding between the unit circle and a graph of trigonometric functions greater than 90 degrees. Their success was limited by their inability to find the *sine* and *cosine* values of special triangles within the circle context. This inability was attributed to a lack of memorization of the values, or a lack of knowledge of the means to find them.

Responding to the disconnect that students have between right angle trigonometry and unit circle trigonometry, Moore (2012) recommends that a student create a drawing that connects a right triangle to the unit circle with the vertex of the angle at the origin and draw a circle with a radius congruent to the length of the hypotenuse. Alternately, the hypotenuse of the right angle could be chosen as the radius of a unit circle leading to the ratios that emerge from using the hypotenuse as the unit of measure for the legs of the right triangle. Conceivably, this would create a connection between the connect students' previous understanding of similar triangles and the unit circle. This method is similar to the method that was deemed too complex and impractical for students' needs in the Kendal and Stacey (1996) study.

Teacher Content Knowledge

As in all disciplines, teacher content knowledge is crucial for students to develop deep conceptual knowledge of trigonometric functions in all of the contexts presented. Shulman (1986) divides teacher knowledge into three constructs: content knowledge, curricular knowledge, and pedagogical knowledge. The challenges to the integration of the triangular and circular forms within trigonometry, and the placement of trigonometry on the trajectory of formal mathematics can be understood within all three constructs. Content knowledge includes not only the content of the curriculum of the course being taught, but also how the content relates to other content on the student's math trajectory. Vertical curricular knowledge, or knowledge of what comes next in the domain, allows teachers to place content in the context in which it will be used, better preparing students for the next course in their path. Strategic pedagogical knowledge is knowledge of where confusion about content within the domain is likely to cause learning difficulties for students. In order for students to thrive, teachers need to have a deep multifaceted understanding of content. No change in curriculum or course content will benefit students without a corresponding increase in teacher content knowledge. The adverse effects of lack of sufficient content knowledge on student learning are well documented and indisputable (Akkoc, 2008).

One measure of vertical curricular knowledge is teachers' understanding of terms that will be used on higher-level trigonometric contexts. Research indicates that there is a lack of consistency in language that teachers use to present the right triangular form of trigonometric functions and the circular form (Moore, 2014). It is no surprise that the lack of coherence of the understanding of the meaning of degree measures and radian measures found in trigonometry students is also present in their teachers. A multiple-case study of six pre-service mathematics teachers (Akkoc, 2007) found that their understanding of the meaning of radian was dominated by concept images of degree, resulting in a reluctance to accept trigonometry functions with real numbers represented as radians. The teachers also held two different concepts of the definition of π: as an angle measured in radians, and as an irrational number. As students, pre-service teachers spent years developing an understanding of degree measure that differs greatly from concept of radian measures of angles. The result is that pre-service teachers demonstrate a shallow understanding of angle measure dominated by degree representations (Thompson, Carlson, & Silverman, 2007). The lack of coherence in the understanding of the relationship between angles and subtended arcs has resulted in the inability of teachers' to conceptualize and transfer the meaning of *sine* and *cosine* from the triangular to the circular form (Moore, 2012).

Teachers will be unable to help students deal with the challenges that the multiple contexts of trigonometric functions present if their own understanding of the relationships between the contexts is weak. Lack of content knowledge regarding the relationship between the angle, right angle, and unit circle contexts in trigonometry teachers resulted in students who had difficulty reasoning about trigonometric functions in a circle context (Moore, LaForest, & Kim, 2012; Thompson, Carlson, & Silverman, 2007; Weber 2005). In a qualitative research study of inservice teachers, teachers of high school students were asked to illustrate the an angle of 30 degrees, then to increase the angle measure. In the resulting drawings the only way the teachers were able to conceptualize the increase in the angle measure was by lengthening the opposite side of the triangle. Consequently they were unable to describe the meaning of the angle measure (Thompson, 2011). Another study of 14 secondary math teachers found that the teachers were unable to think in terms of $cos 100°$ because they could not move beyond the angle restrictions present in the right triangle form (Thompson, Carlson & Silverman, 2007).

Teachers' ability connect new content to both students' previous learning experiences and the future challenges they will face on the trajectory of formal mathematics is made more difficult when the content of the textbooks that they use lack the necessary linkages. Trigonometry textbooks from the turn of the twentieth century used algebra and geometry to prove the trigonometric identities and ratios presented in the right triangular form (Van Sickle, 2012). A 2010 study (Byers) found significant gaps in between the materials presented in high school trigonometry textbooks, level of trigonometry understanding needed by college students. Missing in the high school textbooks were theoretical foundations that are necessary for trigonometry to be applied in higher-level mathematics.

Courtney Baker: I wonder how the use of online textbooks impacts student/teacher knowledge and understanding.

Discussion

According to Vygotsky's theory of the zone of proximal development (Kozulin, Gindis, Ageyev, & Miller, 2003), our ability to understand and use novel information relies on our ability to form connections to our existing set of knowledge and experiences. The way in which students create meaning when they learn new concepts has a great impact on their ability to synthesize later learning that is dependent upon the newly acquired knowledge (Thompson, 2008). The fractured way in which trigonometry is currently taught does not allow teachers or students to create meaningful connections to prior knowledge or to build the conceptual framework necessary for a deep understanding of the multiple contexts of the functional forms of trigonometry. This leads us to two vantage points from which to consider the teaching and learning of trigonometry: the first is trigonometry's place in the trajectory of a student's formal mathematics development, the second is how to create coherence in a subject that is divided into two seemingly unrelated parts.

Courtney Baker: Creating connections is a theme throughout the paper. Starting and ending with the same idea provides a circular argument/purpose that highlights your goals as an author.

The original effort to simplify trigonometry into the ratios of the right triangular form was well intentioned (Van Sickle, 2012). The unintended consequences of stripping the functions of their foundational meaning have brought us to a place where both student and teacher knowledge is quite often limited to a procedural understanding of concepts (Cavey & Berenson, 2005). The simplified representation of trigonometric functions as decontextualized ratios does not allow for a depth of understanding that has meaning when the circular form of trigonometric functions is taught at a later time (Yigit, 2014).

Trigonometry is generally presented to geometry students deep within the chapters of their textbooks in the units that cover triangles. Though early textbooks (Van Sickle, 2012) advocated the use of memorization techniques for trigonometry students to use to solve triangles, these techniques were only introduced after a thorough discussion of the development of the relationships. Rather than connecting student understanding of the properties of triangles they have been studying to the development of the ratios of trigonometric functions, students are taught to use SOH-CAH-TOA to solve right angle side lengths and angle measures. Punching numbers into a graphing calculator will result in satisfactory answers for tests and quizzes, but the connection necessary to ground their understanding in prior knowledge is missing. The presentation of the 30-60-90 and 45-45-90 special triangles seems arbitrary in modern textbooks, but can be connected to the presentation of special triangles with trigonometric ratios in college textbooks from the late nineteenth century (Van Sickle, 2012). The current lack of integration of the understanding of how special triangles relate to trigonometric functions is a missed opportunity to connect this level of trigonometry to students' prior use of the application of the Pythagorean theorem to solve triangles. Using the Pythagorean theorem to derive special triangles would connect students to prior learning, and bring that learning forward to use as a tool in a new environment. Instead, students memorize the ratios of special triangles and the opportunity for deep connected learning is missed. The result of the approach is evidenced in a case study (Cavey & Beerenson, 2005) of a pre-service secondary mathematics teacher that found that her conception of mathematics was procedural based: a collection of definitions and skills to solve problems.

Research suggests that using multiple representations in teaching and learning supports the development of mathematical understanding, and that the ability to connect different repetitions is a strong indicator of knowledge and ability (Byers, 2010). A deep understanding of special triangles as a part of the trajectory of the acquisition of trigonometry would also allow students to form a solid connection to the circular form of trigonometry that is taught in algebra 2 or precalculus. On the unit circle, the reference triangles drawn from the terminal points on the unit circle form 30-60-90 and 45-45-90 triangles. The ratios of the sides of previously encountered special triangles represent the magnitude of horizontal and vertical displacement of the terminal point from the origin that form the coordinate pairs of the unit circle. What is currently taught as an exercise in memorization has the potential to be used as bridge to connect the multiple forms of trigonometric functions. If ideas are sequenced they provide a foundation for learning other ideas that build a network of ideas. Studies highlight that attention must be given to the ideas that trigonometry is about (angle measures, unit circle, right triangles) if students are to do more than simply answer questions using memorized procedures (Moore, LaForest, & Kim, 2012).

The integration of mathematical knowledge is necessary in order to extend mathematical understanding into previously unknown areas (Cavey & Berenson, 2005). A case study (Cavey & Berenson, 2005) that used an integrated approach to develop a teachers' approach to teaching right triangle trigonometry resulted in an increase in the teacher's increased understanding of right triangle trigonometry. The discontinuity of the way that trigonometry is taught in the United States can only serve to magnify the difficulty for both teachers and students to close the cognitive gap that exists between the two forms of trigonometric functions. This is particularly evident in the confusion and lack of competence in students and teachers about how angles are

Courtney Baker: I know why you cited the study twice – but as I read, the citation takes away from the point. Is there a way to connect these two thoughts in one sentence – or make it so that the citation is only used once?

measured. One possible solution would be to introduce the concept of the measure of an angle as a magnitude of radians in the geometry curriculum. Early familiarity would ease the transition from degrees to radians when students are already struggling with the new presentation of trigonometric functions in circular form.

A more drastic change would be to reverse the current triangle-circular sequence of the teaching of trigonometry. Bressoud (2010) and Van Sickle (2012) argue that there is an advantage for students to discover trigonometry in the same way it was discovered in history. This is in many ways a logical approach to teaching trigonometry given that the unit circle representation of trigonometric functions contains the elements used to derive the trigonometric ratios that are presented in the triangular form. Learning trigonometry by building the unit circle with the tools of trigonometry that students have recently acquired would allow them to see the functional nature of trigonometry imbedded in the relationship of line segments to arc length. Bressoud states that this is the form that students are more likely to encounter to model periodic phenomena in the sciences. As previously discussed, special triangles are naturally integrated into the circular form and their current lack of relevance to the SOH-CAH-TOA ratio environment would no longer cause difficulties that lead teachers and students to rely on memorized constructs.

One last option for creating an environment that would help teachers and students integrate all of the contexts of trigonometric functions would be to teach it as a continuous integrated subject. The current manner in which we teach trigonometry to high school as a subject divided into two elements and embedded into separate classes in separate years with different teachers almost assures that students and teachers will have difficulty perceiving trigonometry as a unified whole, no matter how well integrated the curriculum.

Implications for Research

Considering the important place that trigonometry holds on the trajectory of mathematics and the development of the sciences, it is surprising that there is so little research on the teaching and learning of trigonometric functions (Demir & Heck, 2013). The research that does exist follows two veins. One set explores the depth of knowledge of teachers of trigonometry, the other looks at how trigonometry is situated on the trajectory of formal mathematics that leads to calculus (Moore, 2014). There is no research on the impact of students' weak conceptual understanding of trigonometric functions may have on their acquisition of calculus and college level mathematics.

The most pressing apparent need is in the development of ways to strengthen and deepen teacher content knowledge and understanding. The current fragmented form of how trigonometry is taught has been in place for some time and it is reasonable to assume that the current apparent shallow understanding demonstrated by teachers is the result of the disconnected learning environment that they experienced when they learned trigonometry. Support to strengthen teachers can only have the effect of deepening current student knowledge. Students who have a solid conceptual understanding of the multiple applications of trigonometric functions will mature into teachers who will be able to give their students a coherent understanding. Comparative studies of current practice and proposed changes would give us insight into possible next steps.

Courtney Baker: YES!!!

The concepts of trigonometric functions are complex. While assessment of student achievement in tests of skill may give us insight into their ability to calculate answers, they do not give us any indication of how students perceive the meaning of the results of their calculations. There have been several qualitative studies that have attempted to find the extent of teachers' knowledge of trigonometry. There are fewer qualitative studies of student perceptions of their understanding. Qualitative studies have the potential of helping researchers find ways to operationalize how students attain deep understanding of trigonometry. This would give us the tools to look deeper into not only the impact of how trigonometry is currently being taught, but also a greater understanding of the long-term impact on students' acquisition of college level mathematics. Because so little research has taken place, there remains much that we do not know.

Courtney Baker: How could you replicate and extend the studies to include more conceptual knowledge or connections? This might be an area to investigate.

Another avenue of potential research would be replicat[illegible]r follow-up studies of existing research. Such studies would allow us to measure the impact of interventions with greater accuracy. There is a need for greater understanding so that arguments can be developed that strengthen and enrich existing practice or refute of existing paradigms.

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