The Learning Trajectory Pathway in Geometry:

The Calculation of Volume of Three-Dimensional Geometric Objects

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If you watch a young child play with unit blocks you will observe them stacking them, lining them up, considering how they fit together and what configurations are most pleasing to the child. How much do these early experiences with solid geometric forms inform a student's later understanding? The purpose of this paper is to explore what curricular elements and approaches contribute to a student's understanding of solid geometry and the ability to understand the concept of and to calculate the measure of volume of three-dimensional geometric forms. The stated purpose of K-12 education is to prepare students for what lies in ahead of them when their grade school years have ended. If that future includes advanced mathematics, than an understanding of solid geometry and an understanding of the concept and measure of volume are necessary for success. When considering the foundational skills that are necessary for a student to understand the concept and measure of volume it is necessary for the student to build the understanding from their experiences and their previ[illegible]learning

[illegible]here are many and varied elements that students must master to arrive at the point where they can successfully understand the concept of volume within geometric solids and the procedures used to calculate volume. The framework of learning trajectories offers a cogent and useful way to consider the understanding and knowledge that a student needs to acquire as they prepare for high school geometry. Learning trajectories are used to define the pathway that a student takes from the known to the unknown (Simon, 1985). Simon defined it as the "trajectory of students' mathematical

Toya Frank: A strong intro! I'm excited to read.

Toya Frank: Subheading about learning trajectories b/c you are moving out of your introduction.

thinking and learning,” (1985, p. 121). Working from a constructivist perspective, he describes learning trajectories as containing the learning goal, the activities, and the thinking and the learning that students engage in. These three elements make up the “hypothetical learning trajectory,” (p. 133), the path by which learning might proceed. The elements are further described as knowledge of the content, representations, materials, activities, and understanding of students’ conceptions.The trajectory is defined as ‘hypothetical’ because the actual trajectory is not knowable in advance.

Clements and Sarama (2009) expanded the theory of learning trajectories to include the developmental progressions of a student ensuring that a sequence is consistent with the natural developmental levels of a student’s thinking and learning. Within this model they place key tasks designed to “promote learning at a particular conceptual level or benchmark in the development progression,’ (p. 84). They refine Simon’s three elements as: the learning goal, the developmental progressions of thinking and learning, and the sequence of instructional tasks. They argue that complete learning trajectories can alter developmental progressions because it opens up new pathways for learning and development. Beyond the interactions between teachers and students, they conceive of learning trajectories within a specific mathematical domain, the “conjectured routh through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain,” (p. 83).

Toya Frank: As I'm reading, I think you need a subheading here to help your reader transition between ideas. You are shifting from LTs to frameworks for Geometric Thinking

Batt[illegible] (2007) describes a student’s ability to understand volume as a series of arrays. He describes five basic cognitive processes that are necessary for a student to understand arrays of squares and cubes: abstraction forming and using mental models, spatial structuring, units-locating, and organizing-by-composites (Battista, 2007). To attain this level of ability a student must have the knowledge and skill that allows them to successfully implement these processes. Though the pathway that leads to the understanding of volume and solid geometry has not been clearly been defined, it is assumed that the learning trajectory that leads to an understanding of three-dimensional solid geometry and volume measure begins first with an understanding of length (one-dimensional space) and area (two-dimensional space). Curry and Outhred (2005) found this sequence demonstrated in a study of 24 first through fourth grade students. In a series of clinical interviews they found that length measurement may be a prerequisite for area measurement, and that students who were successful in their understanding of area measurement were more likely to be successful at measuring volume by packing layered arrays.

The process of a students acquisition of the foundational geometric concepts of length and area were defined in the van Hiele level theory developed by Diana van Hiele-Geldof and her husband Pierre Marie van Hiele in 1957 (Usiskin, 1982). The model divides geometric concepts into five hierarchical discontinuous levels of understanding:

1. The student can learn the names and shapes of figures as entities recognized by their appearance;
2. The student can identify characteristics and classification of shapes;
3. The student can deduce properties of a figure and recognizes classes of figures;
4. The student understands the roles of postulates, theorems, and construct proofs.
5. The student is able to make abstract deductions. (Usiskin, 1982) .

Readiness for geometry instruction is achieved by assessing a student's current level and providing interventions to move them through the sequence. Attainment of level 4 is necessary for success in geometry. If there is a mismatch of student level and level of instruction, learning may not occur. (Usiskin, 1982). The theory states that a student must pass through the levels in order. Though these are conceived as developmental levels, van Hiele believed that development could be accelerated by instruction. This is similar to the argument that Clements and Sarama made that learning trajectories can alter development by opening new pathways.

Toya Frank: Subheading about CCSSM

The van Hiele levels are clearly evident in the trajectory of geometry that is laid out in the Common Core State Standards (CCSSO/NGA, 2010). Beginning in kindergarten, the standards include the analysis and identification of two- and three-dimensional shapes. The standard in grades 1 and 2 is for students to identify the attributes of shapes. Grade 3, 4 and 5 standards include an understanding of the categories of shapes.

The calculation of volume appears in the 6th grade standard as the formula for the volume of a right rectangular prism in 6th grade (through packing and multiplying edge lengths). Right prisms are added in the Grade 7 standard. The formulas for the volume of cones, cylinders, and spheres are added in grade 8 with little or no connection with earlier concepts.

The Progressions for the Common Core State Standards in Mathematics (Common Core Standards Writing Team, 2016) provide detail of the progression of geometry across grade levels, explaining the reasoning behind the given sequence. Within the detailed progression for geometry the discussion of volume is first addressed in the 7th grade narrative. Dissection of figures is recommended for calculating volume. Explanations for volume of cylinders, cones, and spheres are not included in the standard. Mention of Cavalieri's principle for calculating the volume of a sphere is given, but no suggestions or recommendations are given for connecting prior knowledge to the principle.

The lack of connection between prior learning and the presentation of procedures for calculating volume are concerning. There is evidence that elementary students use area and volume formulas without understanding them (Dorko & Speer, 2013). This problem has not been sufficiently addressed in the current CCSS standards and actually may be exacerbated by the tacit assumption that the explanations of the formulas for the volume of cylinders, cones, and spheres are beyond the ability of 8th grade students' comprehension (Common Core Standards Writing Team, 2016).

Toya Frank: suggestion: subheading about challenges to understanding volume/3d objects

The van Hiele theories are traditionally applied to plane geometry, and as a foundational theory behind the trajectory of geometry in the CCSS (Howse & Howse, 2014) may explain the breakdown of the learning trajectory pathway to three-dimensional geometry. Exploring how the van Hiele levels relate to three-dimensional geometry, Gutierrez (1992) applied the van Hiele levels to the context of student use of three dimensional manipulatives. He defined level one as a global perception of the shapes of geometric solids, level 2 as the examination of the differences in geometric solids apparent through observation, and level 3 as perceived comparison of the mathematical properties of the geometric solids. Students who have attained level 4 are able to analyse solids prior to manipulation with a high ability of visualization of transformations. Though these are logical extensions of van Hiele's' theory into three-dimensions, there is no indication that they have been integrated into existing standards.

Elementary students' struggles with volume are well documented (Battista & Clements, 1998; Sisman & Aksu, 2016). Students' difficulties in understanding volume measure include confusion in differences between two- and three-dimensional figures, misunderstanding of the enumeration of the faces of an array, and confusing surface area and volume formulas (Sisman & Ask, 2016). Early experience with isolated procedural understanding of volume as presented in the CCSS cannot provide the conceptual understanding necessary direct them away from these misconceptions.

The same difficulties presented by elementary students were found in a study of college calculus students' approaches to volume measure and calculation. (Dorko & Speer, 2013). As in the research on grade school students, the findings indicated that some college students' incorrectly calculated surface area instead of volume because they incorrectly believed that areas of faces measured three-dimensional space. In addition, errors in student recall of memorized formulae also contributed to incorrect calculations of volume.

The confusion about the concept of volume and the proper use of procedures that the college calculus students displayed were also apparent in a study of 445 sixth grade students' misconceptions of length, area, and volume measure (Sisman & Aksu, 2016). The study found that both conceptual and procedural knowledge of the students in all three areas was very shallow. Specifically as dimensions increased, errors increased. The errors in volume measurement included miscounting units, counting faces or only visible units of the cubes. Lack of strength in both procedural knowledge and conceptual understanding contributed to the misconceptions.

Volume measure is only a single topic on a student's trajectory toward more advanced mathematics. Perhaps it is easier to see the pathway by considering the perspectives of students who are successfully applying the understanding of volume in mathematics beyond high school. Dorko and Speer (2013) surveyed 198 differential calculus students and conducted 20 clinical interviews with a subset of students. They found that students who thought of volume in terms of *area of base times height* or an array of layers were more successful in calculating volume. They recommend that instructors provide students with opportunities to model volumes with arrays and to to connect the models with the formulas.

It would be easy to assume that the difficulties in understanding volume measure that students demonstrate is a function of their developmental readiness. After all, Curry and Outhred (2005) found that students in fourth grade were much more successful than first grade students were at understanding volume by packing. But, the misconceptions found in 6th grade students (Sisman & Aksu, 2016) were also present in college calculus students (Dorko & Speer, 2013), It is safe to assume the calculus students successfully completed high school geometry. Battista (2007) found that an understanding of arrays was necessary for a student to successfully understand volume measure. This was demonstrated by the success of the college calculus students who employed arrays in their concept of volume (Dorko & Speer, 2013).

Returning to Clements and Sarama (2004): Their perception of learning trajectories as the process of moving students through a developmental progression of

Toya Frank: suggestion: subheading about technology and geo thinking

levels that support achievement within a mathematical domain challenges us to find a way to help students make the leap from two-dimensional geometry to three-dimensional geometry.

Though two-dimensional objects are easy to render on a page or in a textbook, three-dimensional objects can only be represented in two-dimensional physical media as abstract illustrations. Even photographs cannot give the illusion of depth necessary to understand the image in three dimensions. New technology gives us multiple means to explore three-dimensional shapes in a virtual environment. Perhaps technology can give us the ability to move students from two-dimensional to three-dimensional understanding.

Dynamic geometry software (DGS) is computer programs designed to enable students to explore geometric concepts by allowing them to manipulate abstract representations of geometric objects (Borges et al., 2016). A meta-analysis (Chan & Leung, 2014) of quasi-experimental studies on the effectiveness of DGS on student's mathematics achievement in K-12 education found that the use of DGS had a significant positive effect of the achievement outcomes of students. While the meta-analysis did not provide information on achievement in volume measure, a study of high school seniors looked at the effects of exposure to computer-assisted instructional packages on solid geometry understanding (Gambri, Ezenwa, & Anyanwu, 2014). The subjects were divided into three groups: one treatment group was exposed to computers with animation depicting the concepts of solid geometry with narration, one treatment group was exposed to computers which displayed animation accompanied on-screen text that supported the concept being presented, and the control group was exposed the traditional lecture method of instruction. Both treatment groups that were exposed to the computer animated software achieved higher mean scores on solid geometry achievement than the control group.

In a meta-analysis of 46 primary studies of K-12 learning environments (Li & Ma, 2010), multimedia computer technology was found to have statistically significant positive effects on mathematics achievement. Not only was the use of computer technology in education in K-12 classrooms found to be effective, the positive effect was found to be greater when combined with a constructivist approach to teaching. Several studies selected for the meta-analysis looked at the impact of virtual manipulatives in the classroom and found positive effects on student achievement.

As evidenced by the meta-analyses referenced in this paper (Chan & Leung, 2014; Li & Ma, 2010) the presence and availability of technology in the classroom has resulted in a large body of research. There appear to be far fewer studies of computer use in high school geometry classrooms. This may be an indication of a dependence on textbooks or other resources in geometry classrooms.

Manipulatives are seldom seen in high school mathematics classrooms and are usually associated with elementary mathematics instruction. Tietze (1992) wrote about a curriculum design for high school students that included manipulatives that she had found to be successful in supporting their understanding of two- and three-dimensional geometry. In spite of apparent paucity of research on the use of technology for the teaching and learning of solid geometry in high school classrooms, there is enough evidence to support the use of DGS to supplement the traditional curriculum materials used in a high school geometry courses. Extending Tietze's design with virtual manipulatives has the potential to connect the pathway of the geometry learning trajectory from one- and two-dimensional geometry to three-dimensional geometry and the calculation of volume measure.

Toya Frank: Heading: Conclusion

Ultimately learning trajectories are about the experience and the success of individual students. While the learning trajectory framework has been applied to the domain of geometry in this paper, a curriculum that is developed from the pathway is only effective if it can be implemented with fidelity to its intent. Clements and Sarama (2009) recommend that learning trajectories should be "reconceptualized or created by small groups or individual teachers, so they are based on more intimate knowledge of the particular students involved," (p. 85) In their concluding remarks they recommend placing the small model within a broad hypothetical learning trajectory that describes the teachers' goals. Simon (1995) too addressed the needs of the teacher when stating that the design of the learning trajectory should include flexibility in the teacher's approach, a deep understanding of the content, and an awareness of students' current level of understanding. Successful implementation can only occur with a will to strive to make the connections from the students' knowledge base to new knowledge.

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