Symbols and Words: The Application and Effect of the Conceptual and Procedural Knowledge Framework

In order to study mathematics understanding and learning, a conceptual framework with definitions and meanings that are clearly understood within the mathematics research community must be established. A well-defined framework allows for meaningful dialogue and understanding across the discipline. Research can be designed around the framework, outcomes can be measured according to the operationalization of the components of the framework, and results can be understood within a shared conceptualization of the components.

Within mathematics education research literature a well-established and often used framework parses mathematical knowledge into two domains: procedural knowledge and conceptual knowledge. This paper will look at the application of the conceptual and procedural knowledge framework in research studies with three different lenses. The first group of studies employs the elements of procedural knowledge and conceptual knowledge as entities. The second group of studies uses an expanded and refined definition of procedural and conceptual knowledge (Star 2005) to examine procedural fluency. And, the final group of studies explores the impact that changes to procedural knowledge and conceptual knowledge may have on each other (Rittle-Johnson, Schneider, & Star, 2015).

The Conceptual and Procedural Knowledge Framework

The conceptual and procedural knowledge framework was defined by Hiebert and Lefevre (1986) in their seminal article to provide a useful way for researchers to understand students' learning processes. They acknowledged that the relationship between procedural knowledge and conceptual knowledge was not well understood, and that often mathematics knowledge may be an inseparable combination of both forms of knowledge. In spite of this, they

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argued that distinguishing types of knowledge would provide a way to understand the failure or success of building mathematics understanding.

Hiebert and Lefevre (1986) define conceptual knowledge as a connected web of knowledge: a network of linked relationships that are as important as the discrete elements. The web of knowledge is conceptual only if the learner recognizes the relationships between elements. Conceptual knowledge not only describes the connections between known elements, it also describes the connections made between existing knowledge and newly acquired knowledge. The context is created by the network of relationships, allowing the learner the freedom to apply existing knowledge to novel problems.

Procedural knowledge as described by Hiebert and Lefevre (1986) is the use of the formal language and the symbolic representation system of mathematics. Algorithms or rules are used to complete tasks with an awareness of only surface features. The knowledge of the meaning of the processes is not accessed or necessary for successful completion of tasks. Step-by-step instructions prescribe how to complete tasks. These tasks may then be sequenced into superprocedures that incorporate lower level subprocedures. Procedural knowledge allows students to solve complex superprocedures as a chain of prescriptions without knowledge of the meaning of the task.

Distinguishing Between Conceptual Knowledge and Procedural Knowledge: Studies One, Two, and Three

Study One

The first study is a comparative study of the conceptual and procedural knowledge of fraction operations of preservice teachers in Taiwan and the United States (Lin, Becker, Byun, Yang, & Huang, 2013). The researchers applied Hiebert and Lefevre's (1986) description of procedural knowledge as knowing “when” and “how” to use procedures appropriately, and conceptual knowledge as “why.” They used Liping Ma’s (1999) Profound Understanding of Fundamental Mathematics (PUFM) was used as the model for the ideal structure of the combination of both procedural and conceptual knowledge.

Methods. The design was a quantitative study using a paper-and-pencil test adapted from a previously validated instrument developed to assess 4th and 5th grade students (Cramer, Post, and del Mas, 2002) to determine teacher procedural and conceptual knowledge of fractions. The test covered addition, subtraction, multiplication, and division of fractions. Procedural knowledge was tested with algorithmic questions. For conceptual knowledge, participants were asked to create models for provided algorithms that could be used for teaching children. The paper-and-pencil test allowed participants to explain their thinking, and researchers to standardize grading. The test was administered the first week of class in the fall to 47 preservice teachers in Taiwan and 47 preservice teachers in the United States.

Results. Comparing the fractional knowledge of Chinese and Americans, the researchers found that overall the Chinese teachers did significantly better than the Americans on all procedural items, and on all conceptual dimensions except division which displayed lower scores for teachers from both countries. There was a weak, but not statistically significant, positive correlation between procedural and conceptual knowledge in addition, subtraction, multiplication, and division of fractions of teachers in both the United States and Taiwan. Findings were similar to several named studies in the article. Procedural knowledge scores exceeded conceptual scores in all dimensions for teachers from both countries.

Critical Comments. The researchers determined that the weak correlation between conceptual and procedural knowledge of fractions suggests that higher procedural knowledge does not cause nor predict higher level conceptual knowledge. Implications for teacher education included the need to develop content knowledge to avoid the poor teaching that comes from a teacher who is insecure in subject knowledge. The researchers also state that though prior international comparisons of mathematics achievement has been focused on grades 4-12, comparative analysis of preservice teacher performance is essential, allowing comparisons between countries and the opportunity for researchers and teachers to learn from each other. Limitations of the study included sample size (97 subjects), that only preservice teaching majors were included, and that the study only covered knowledge of fractions.

Implications. Implications for teacher research included the need to develop content knowledge to avoid the poor planning and teaching that comes from a teacher who is insecure in subject knowledge. Also, analysis of teacher knowledge is necessary for assessing competence. International assessments also allow for comparisons between countries and the opportunity for researchers and teachers to learn from each other.

Study Two

The second study (Rayner, Pitsolantis, & Osana, 2009) analyzed the relationship between procedural and conceptual knowledge of fractions and mathematics anxiety in preservice teachers. The authors cited Hiebert and Lefevre (1986) when defining procedural knowledge as recalling and carrying out specific steps to solve problems and conceptual knowledge as understanding the mathematical principles that underlie the procedures.

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Methods. Thirty-two undergraduate preservice teachers knowledge of fractions were assessed using a validated paper-and-pencil test designed to assess upper elementary students' procedural and conceptual knowledge of fractions with an additional 2 researcher-developed items to test conceptual knowledge. The additional questions assessed participant procedural knowledge by presenting a equation to solve, and assessed conceptual knowledge by asking the participant to create a world problem that placed the equation into a real-world context. Researchers used the Revised Mathematics Anxiety Rating Scale (RMARS, Baloglu, 2002) developed for undergraduate and graduate students to measure mathematics anxiety in the participants. The resulting data allowed for a quantitative analysis to determine the relationship between knowledge of fractions and mathematics anxiety.

Results. The researchers found that preservice teachers scored higher on procedural knowledge than conceptual knowledge. Seventy-two percent of the participants scored in the middle range for anxiety, 19% were in the low range, and 10% were in the high range. When comparing knowledge scores with anxiety results, researchers found that there was a significant negative correlation between both procedural and conceptual knowledge and anxiety. Greater anxiety correlated with fewer correct responses.

Critical comments. This study replicated the negative relationship between mathematics anxiety and performance on complex procedures found in previous studies of undergraduate psychology students. The study adds to the body of literature on the relationship between content knowledge and mathematics anxiety and is the first to examine the role of mathematics understanding in preservice teacher anxiety. The researchers surmised that the negative correlation between conceptual knowledge and anxiety is consistent with previous studies and may indicate that conceptually based instruction facilitates a more meaningful understanding of mathematics. The researchers determined that procedural proficiency should also be addressed. Limitations included small sample size (32 participants), limited scope (fractions only). The design of the study did not allow for the exploration of causal relationships. Limitations include small sample

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size (32 participants), limited scope (fractions only). Gender differences were not explored but were noted by researchers. The design of the study did not allow for the exploration of causal relationships.

Implications. This study replicated the negative relationship between mathematics anxiety and performance on complex procedures in previous studies of undergraduate psychology students. The study adds to the body of literature on the relationship between content knowledge and mathematics anxiety and is the first to examine the role of mathematics understanding in preservice teacher anxiety.

Study Three

The third study (Cheng-Yao, 2010) used tests of preservice teachers' procedural and conceptual knowledge of fractions to determine if web-based instruction is more effective than traditional instruction. This study employed Hiebert and Levre's (1986) definition of procedural knowledge as knowing how to solve mathematics problems, and conceptual knowledge as knowing why a procedure should be used.

Methods. This study was a pretest/posttest experimental design with forty-eight elementary education math methods students randomly assigned to complete a 6-week fractions unit delivered by the same instructor with either traditional instruction or web-based instruction. The instrument used to assess knowledge was adapted from Cramer et al. (2002) and Ma (1999) and modified by the researcher to provide more emphasis on procedural and conceptual knowledge. The assessment of procedural and conceptual knowledge was conducted in a manner similar to Study 2 above: Procedural knowledge was assessed with the presentation of an equation to solve, and conceptual knowledge was to be demonstrated by the participant creating a story or model for the equation. A quantitative analysis was conducted on the test scores.

Results. The pretest scores for procedural knowledge were higher than conceptual knowledge. Using the definitions drawn from Hiebert and Lefevre (1986) the researcher determined that the results indicated that participants could *do*, but could not explain *why*. Post test scores on web-based instruction were significantly higher than for traditional instruction. Conceptual scores were also significantly higher for online class instruction than for traditional instruction but remained lower than procedural scores.

Critical comments. Results suggest that web-based instruction is efficient as indicated in earlier studies. This may be because of the student-centered approach, room for student exploration, and immediate feedback. Limitations of the study included the short span of time covered which meant that retention of knowledge was not assessed. Additionally, the study covered only knowledge of fractions, had a small sample size (48 participants), and only used elementary education majors. These factors limit generalizability of the results.

Implications. Results suggest that web-based instruction is efficient as indicated in earlier studies. This may be because of the student-centered approach, room for student exploration, and immediate feedback.

Comparison of Studies One, Two, and Three

All three studies chose to look at fractional knowledge in preservice teachers because of the importance of fractional knowledge, the complexity of the topic, and difficulties that preservice teachers reportedly have had with knowledge of fractions. A comparison of the results across all three studies reveals procedural knowledge scores that are higher than conceptual knowledge scores using the lens that the researchers chose to define procedural knowledge. All three studies (Cramer, Post, & del Mas, 2002; Rayner, Pitsolantis, & Osana, 2009; Cheng-Yao, 2010) used purely quantitative analysis which did not allow for feedback by participants on their thinking processes as they solved problems. Consequently it is impossible to determine whether the participants made connections that were not demonstrated in the test results: whether they solved problems by rote processes or made choices driven by an underlying understanding of the concepts involved (Star, 2005). The nature of how the researchers used the conceptual and procedural knowledge framework was not the same across the studies. It was clear the researchers used from each what they thought would be most applicable to the needs of their study. Using this framework, consistently, procedural scores were higher than conceptual scores whether assessing for comparison to other groups (Lin et al, 2013; Cheng-Yao, 2010) or assessing for comparison to another variable (Rayner, Pitsolantis, & Osana, 2009).

In this group of studies procedural knowledge was operationalized as the ability to solve algorithmic problems, and conceptual knowledge was demonstrated by the ability of the participants to verbalize their understanding. This approach was reflected in study two (Rayner, Pitsolantis, & Osana, 2009) and three (Cheng-Yao, 2010), requiring the participants to demonstrate conceptual knowledge in a language rich environment. In their definition of conceptual knowledge Hiebert and Lefvre (1986) do not in any way define conceptual knowledge as the learner's ability to verbalize mathematical constructs. When discussing the difference between procedural and conceptual knowledge they describe story problems as conceptual and the number sentences as procedural, not as definitive, but as illustrative.

Verbal articulation of mathematical concepts may demonstrate conceptual understanding, but does the absence of a verbal response imply lack of conceptual understanding? The somewhat simplistic application of Hiebert and Lefevre's (1986) framework in these studies does

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not take into account the intersection and influence that one type of knowledge may have on the other. For all three studies the ability of the participant to solve equations was used only to assess procedural knowledge even in the first study where participant wrote explanations of their processes next to their solutions. Though all three studies used pencil-and-paper tests, there was no provision to gather data on participants' conceptual understanding that may have been embedded in their procedural work. When conceptual understanding was only tested by asking participants to create scenarios that would embody given mathematical expressions, conceptual theoretical mathematical understanding that might have been present but not applicable in a real-world context was not accessed or assessed.

Knowledge Type and Quality: Studies Four and Five

In a response to Hiebert and Lefevre's (1986) bifurcation of mathematical knowledge into conceptual and procedural types, Star (2005) redefined conceptual knowledge as knowledge of concepts, principles, and definitions; and procedural knowledge as knowledge of procedures, algorithms, and the sequence of steps used in problem solving. He argues that contrary to the common perception of procedural knowledge as superficial and rote and the perception that conceptual knowledge is knowledge that is known deeply, that both types of knowledge can be either superficial or deep or anything in between. He describes the depth of knowledge as *knowledge quality.*

Using this alteration of the conceptual/procedural framework, the following two studies focus on flexibility in procedural knowledge. Researchers in both studies interpret flexibility in procedural processes, or knowledge of multiple strategies, as an indication of deep procedural knowledge. Looking at procedural knowledge without reference to conceptual acknowledges the importance of procedural knowledge as valuable in and of itself (Star, 2005). As in the prior three studies, the participants of the first of these two studies were undergraduates. The participants in the second study were grade school students.

Study Four

The fourth study (Maciejewski & Star, 2016) was a teaching intervention designed to promote flexibility in procedural knowledge in first year undergraduate calculus students. The researchers sought to determine not only if procedural flexibility could be developed, but also if it resembled expert-like procedural performance.

Methods. The design was quasi-experimental design using a pretest/post test. Two sections of an introductory calculus course taught by the same instructor were selected for the quasi-experimental study. A pretest on differentiation was given. After receiving a lesson that contained a traditional sequence of instruction on procedures, the control section was given a typical worksheet for homework. The treatment section was given a worksheet that specified two approaches for solving each assigned problem and were asked to describe which method they preferred. The homework assignment was followed by a re-administration of the pretest as a posttest.

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Results. There were no significant differences in the sections' score averages. Both groups' scores improved on the posttest. The treatment group used a greater variety of strategies than the control group. Many students chose to use processes that took longer to solve because of familiarity with the form.

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Critical comments. Even though some students chose to use longer processes, the researchers determined that the treatment group moved closer to expert-like performance. The researchers felt that as novices the students had not yet developed adequate problem classification schema and that they would become more efficient with more experience. A limitation of the study is that researchers did not explore the sociomathematical norms of the classroom and there is a chance that student choices were driven by a perception of teacher expectation. Also, students were not randomly assigned to the treatment group, though the researchers determined that the two sections were fairly homogeneous.

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Implications. The authors conclude that it is possible to use an instructional task to support the development of undergraduate students' flexible use of procedures. Because the control group did not demonstrate flexibility after practice, the authors determined that an activity that prompted critical reflection by presenting tasks that prompt students to resolve questions in different ways and allow for the comparison of different solutions may allow for the development of deep procedural knowledge.

Study Five

Purpose. The 5th study (Lamb, Bishop, Philipp, Whitacre, & Schappelle, 2016) used clinical interviews to investigate the relationship between student flexibility in procedural problem solving and mathematics performance in students grades 2, 4, 7, and 11. The researchers sought to determine the degree to which flexible ways of reasoning influenced performance on integer problems. The wide grade span was chosen to cover a wide range of student learning experiences: from those who had not yet received school-based integer instruction to students who were enrolled in precalculus or calculus courses and therefore deemed to be successful high school mathematics students.

Methods. Individual clinical interviews were conducted and videotaped at the students' school sites. The interviews were standardized and all students were asked to complete the same 25 open number sentences. Interviews were coded both for underlying reasoning and for correctness. Five broad categories and 41 sub-codes provided detail into students' specific

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strategies. The five categories broken down into ways of reasoning: order-based, analogy-based, computational, formal, and developmental. Flexibility was measured by the variety of ways that students used to solve integer-arithmetic tasks. Proficiency with a particular way of reasoning was demonstrated when a student used it three or more times. The number of ways that students used ways of reasoning that they were proficient in was the measure of flexibility.

Case studies were performed on three 7th grade students who exemplified the relationship between flexibility and accuracy. The first student chosen completed 32% of the open number sentences correctly. The second student completed 64% of the problems correctly. The third case completed 100% of the problems correctly.

Results. Seventh graders had the greatest spread in flexibility. Eighty-five percent of 11th graders used 3 or 4 methods. Flexibility scores correlated positively with performance. This result held across the case studies. The first student and second cases used one type of reasoning almost exclusively which limited their options for solving problems. Though each of them focused on *different* forms of reasoning, the fact that each student appeared to have one way of reasoning appeared to negatively influence success. The third case, who had completed every open number sentence correctly, flexibly used a wide range of strategies on the problems and appeared to choose strategies that corresponded with the underlying structure of the sentence.

Critical comments. Case studies provided insight into the relationship between flexibility and performance on open number sentence problems. Across all age groups the correlation between flexibility and performance held.

Implications. The authors conclude that students who rely on a single way of reasoning may be impeded in their success because of their limited flexibility and that multiple ways of reasoning promotes successful performance. For every participant group the correlation between flexibility and accuracy held; more flexible students were more successful.

Comparison of Studies Four and Five

Though study four (experimental teaching intervention; Maciejewski & Star, 2016) and study five (mixed method clinical interviews; Lamb et. al., 2016) addressed flexibility in procedural knowledge, the approaches were very different. Though neither study used the language of the conceptual and procedural knowledge framework, both studies explored the impact of deep procedural knowledge (Star, 2005) exemplified by procedural flexibility. Depth of conceptual knowledge was implied in study four (Maciejewski & Star, 2016) within the discussion of students' process of which method to use to solve problems for those students who used multiple methods. In study five both the clinical interviews and the more in-depth case study interviews provided insight into the procedural and conceptual understanding of the participants. Participant responses allowed the researchers to determine not only the presence of reasoning types but also the manner in which they were chosen. The interview process uncovered participants' procedural and conceptual knowledge, whether solutions were pursued through rote processes or driven by connected knowledge (Hiebert & Lefevre, 1986) and the depth or lack of depth of the participants' procedural and conceptual knowledge (Star, 2005).

The Relationship between Procedural Knowledge and Conceptual Knowledge: Studies Six, Seven, and Eight

There is a tacit belief in the mathematics education research community that conceptual knowledge is a higher, more valuable form of knowledge. This is not borne out by Hiebert and Lefevre (1986). Though the oft cited article may be responsible for the division of mathematical knowledge into two distinct types, the authors discusses at length the relationships between

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procedural knowledge and conceptual knowledge arguing that competence in mathematics relies on the "significant, fundamental relationships between conceptual and procedural knowledge." (paragraph 32).

In a review of the literature on procedural and conceptual knowledge (Rittle-Johnson, Schneider, & Star 2015) the authors explored the relationship between the two types of knowledge. They found that the broad agreement that conceptual knowledge supports procedural knowledge has resulted in ample research on the conceptual-then-procedural sequence of instruction. Consequently, though there is evidence that relationships between the two forms of knowledge are often bidirectional there has been little research on the effect of presenting procedural instruction before teaching conceptual understanding. The authors conclude that "the belief that procedural knowledge does not support conceptual knowledge is a myth." (p. 594)

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The following studies explore the effect of sequencing instruction on the development of conceptual knowledge and procedural knowledge.

Study Six

Purpose: The sixth study **(**Rittle-Johnson & Alibali, 1999) was a quasi-experimental study of fourth-grade students designed to provide causal evidence about the relations between children's conceptual and procedural knowledge of equivalence by examining the impact of instruction on the concept of equivalence on problem-solving procedures and the impact of instruction of problem-solving procedures on conceptual understanding of equivalence. The researchers hypothesized that increasing children's conceptual knowledge would lead to gains in procedural ability and that gain in procedural knowledge would lead to gains in conceptual understanding.

Method. A paper-and-pencil pretest was given to the students to identify and group the students by whether or not they were able to solve standard equivalence problems correctly. Students who solved the problems incorrectly were randomly assigned to either conceptual instruction, procedural instruction, or no instruction (control). The conceptual-instruction group instruction consisted of presentation of a problem and then being told that the amounts before the equals sign needs to equal the amount after it, meaning that the numbers need to add up to the same amount on both sides. No instruction was given on procedure. The procedural-instruction group was presented with a problem and the children were taught grouping procedures to solve. Two cycles of lessons followed by assessments were given to both instruction groups. A posttest of problems identical to the pretest was administered. Performance on transfer of knowledge as a measure of conceptual knowledge was assessed by testing student performance on unfamiliar procedures.

Results: Children who received instruction improved more than children who did not receive instruction. Most children from both instructions groups used correct procedures on the posttest. Procedurally instruction children used the instructed procedure on all posttest problems. Children from the conceptual-instruction group used multiple types of procedures, confirming that the conceptual instruction did not directly teach a specific procedure. Children who received procedural instruction increased their conceptual understanding. Children in the two groups solved an equivalent number of problems correctly on the posttest.

Critical Comments. The authors state that the use of a pretest/posttest design provided causal evidence that conceptual and procedural knowledge influence one another. The use of repeated assessments on conceptual and procedural knowledge allowed for the detection of gradual changes in children’s understanding. The intersection of the two types of knowledge highlighted the causal, bidirectional relations between conceptual and procedural knowledge. This relationship may not be symmetrical. Gains made by the children who received conceptual instruction were greater than gains made by children who received procedural instruction. Though evidence suggests that the two types of knowledge influence each other the mechanisms of *how* still need to be explored.

Implications. Children who were taught procedurally did not attempt to solve problems with minor variations in surface features. These findings suggest that children may benefit most from conceptual instruction that helps them to invent correct procedures on their own. It is interesting to note that the conceptual instruction in study 6 was an explanation of the meaning of the equals sign. No real-world connections were made and all of the instruction that the children received was represented in the presented equation.

Study Seven

The seventh study was an experimental study that explored the effect of organizing arithmetic fact practice around equivalent practice (McNeil et al., 2012) on children ages seven through nine, examining the structure of early input and the role it may have in shaping children's understanding of fundamental mathematics concepts (operations on the left of the equals sign, equals sign means to *do* something). The authors chose to study the concept of mathematical equivalence because they state that it is "one of the most fundamental concepts in mathematics." (p. 11). The authors hypothesized that arithmetic practice that is organized by equivalent sums will lead children to construct a better understanding of math equivalence than practice that is not organized by equivalent sums.

Method. The design was a posttest-only randomized experiment. Randomization occurred at the individual level eich child randomly assigned to the condition of equivalent

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values with practice problems grouped and presented by equivalent values, the condition of a shared addend with practice problems grouped iteratively by shared addend, or no extra practice over and above ordinary school homework. Three practice sessions one-on-one with a tutor were given with paper-and-pencil homework assignments between each session. A posttest was given to assess their understanding of equivalence and computation fluency. Children in the no practice condition were then assigned to either of the two practice conditions. This subset of children participated in a randomized experiment with a pretest, intervention, and follow up. Finally all children receive post-test assessment. The larger experiment was a posttest-only randomized experiment, with random assignment of individuals. The sub-experiment was a pretest-intervention-posttest design with random assignment of individuals. Posttest assessment to all children to measure their understanding of math equivalence consisting of equations solving math equivalence problems, equation encoding by reconstructing math equivalence problems after viewing for a set period of time, and defining the equal sign by responding to a set of questions about the name and possible meanings for the equal sign.

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A follow-up assessment of the equivalence problems was given with the children instructed to make each side of the equal sign the same amount. If the child gave the correct answer the tutor gave positive feedback. If the child provided an incorrect number the tutor, using a script that emphasized the equal value of each side, provided the child with the correct answer. The purpose was to examine practice conditions on children's openness to learn from brief instruction.

Results. Children's understanding of math equivalence was poor overall with most children solving zero or all four equations correctly. Children in the condition of equivalent values demonstrated better understanding in both problem solving and encoding performance than children in the other two conditions. Children of the equivalent values condition were more likely to define the equal sign relationally. Authors state that the results that organizing arithmetic facts into conceptually related groupings may improve children's understanding of mathematical equivalence. Performance on the follow-up assessment was still low with children in the equivalent values condition more likely than children in the iterative condition to solve correctly.

Children in the pretest-intervention-follow up group who received the equivalent values practice were more likely than those who received the iterative practice to solve equations correctly. These children performed significantly better than the children who were not given the pretest. The authors state that this result suggests that practice organized by equivalent values may provide more benefit to children who have been previously exposed to math equivalence problems.

Critical comments. Pretest measures of understanding were not gathered on the full sample to avoid pretest sensitization: that simple exposure to problems would improve performance later tests of math equivalence. The authors argue that the similar, but stronger results of the equivalent values condition over the iterative condition from the pretest-posttest group supported the larger design.

Implications. The authors speculate that the organization of procedural practice around equivalent values may help children build conceptual knowledge and lead to improvement in conceptual understanding of procedures because structurally redundant examples may increase children's chances of inducing arithmetic principles, build familiarity with addend pairs that share the same sum, or increase children's noticing of transitive relationships. "Practical modifications to children's early learning environments can affect their understanding of foundational mathematical concepts" (p. 1119)

Study 8

The eighth study was an experimental mixed methods study of 72 seven and eight year olds that investigated the effect of problem solving practice on learning (Canobi, 2009) and the interactions between conceptual and procedural knowledge. The study was designed to assess whether there is an iterative relationship between conceptual and procedural knowledge. Procedural knowledge is defined as the skills required to solve individual mathematical problems. The author states that exploring children's self-reported procedures with evidence of their problem-solving accuracy provide an accurate characterization of their procedural skills. Conceptual understanding is defined as knowledge about the underlying unifying principle of the structure of the problem. This definition leads the author to determine that assessing knowledge of mathematical principles or mathematical laws as expressed by children. Implicit conceptual understanding though will not be made evident by children's verbal reports. The author expected that highlighting conceptual relationships between problems through sequencing would increase procedural skill. Additionally, it was predicted that executing procedures to solve problems would improve children's report on conceptual relations between problems. It was anticipated that children's initial procedural skills would predict the level of conceptual advances made as a result of procedural practice.

Methods: Children were randomly assigned to either a conceptually sequenced condition, a randomly ordered condition, or a no-practice condition. Procedural and conceptual knowledge were tested before and after a practice phase. Participants were given a computer-based problem-solving task pretest of whole number addition and subtraction problems that

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allowed the children to view the previous problem with the correct answer as they solved each problem. A second group of test problems consisted of children watching a puppet solve a problem using counters, and then responding to an interviewer about whether the puppet could solve an additional problem using the same counters. After each problem the children were given feedback pointing out the nature of the relationship between the current problem and the previous problem. Children judged the commutative, subtraction complement, or identity relationship with the previous problem.

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Practice was given randomly. Children received either conceptually sequenced worksheets, randomly ordered practice, or no practice as a control group. The sequenced problems were presented in conceptually sequenced pairs. This was expected to maximize opportunities to notice conceptual relations between problems as they executed problem-solving procedures. The randomly ordered practice condition was to provide equivalent practice with less obvious conceptual relationships. The no-practice control group completed nonmathematical worksheets. A posttest with identical procedures to the pretest was given. Children's reporting on conceptually sequenced problems was used as a conceptual measure and reporting on randomly order problems was used as a measure of procedural knowledge.

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Results: Procedural practice of randomly sequenced problems improved accuracy. Conceptual sequencing of practice problems enhanced children's ability to extend their procedural skills into new unpracticed problems. In addition, well-structured practice led to improvement in children's ability to identify and report on conceptual relationships between problems. As anticipated, the initial levels of procedural knowledge predicted the conceptual knowledge advances that the participants made. These findings suggest that there is an iterative process between the development of both conceptual and procedural knowledge in children's addition and subtraction skills. The pairing of practice problems appeared to allow the participants to consider the relationship between the problems. The experience of solving practice problems that were ordered to make underlying concepts more obvious appeared to allow the children to apply their improved problem-solving skills to new unpracticed addition and subtraction problems. In comparison, the children who completed worksheets with randomly ordered problems only increased their scores on previously presented problems.

Critical comments. Children were able to verbalize key addition and subtraction concepts even though the intervention only involved asking the children to solve problems using their preferred problem-solving procedures. The author found that children’s initial procedural skills influenced their ability to make conceptual inferences as a result of procedural practice (iterative cycle). The author argues that when children know answers to problems they have a greater capacity to notice the principles that link the problems together, especially when the problems are presented in ways that make the links overt. Because these children do not need to focus on computational requirements, they are able to make discoveries about the structure of the domain. This study was limited to young children and addition and subtraction processes.

Implications. The author argues that the study’s findings support an iterative account of children's basic addition and subtraction development. Conceptual relations helped the children to extend their procedural learning beyond problems they have already solved to new problems. Additionally, the changes in children’s reportable conceptual understanding in the children who were given procedural practice suggests that reportable conceptual knowledge can be developed as a result of experiences that build procedural understanding. The author states the need for the exploration of how children’s conceptual and procedural knowledge influence each other.

Comparison of Studies Six, Seven and Eight.

Of the three studies, the first (Rittle-Johnson & Alibali, 1999) presented the most compelling evidence of the possible iterative nature of the supportive interplay between conceptual knowledge and procedural knowledge. All three articles defined conceptual knowledge as an understanding of the mathematical principles present in presented equations. The authors of studies six (Rittle-Johnson & Alibali, 1999) and eight (Canobi, 2009) describe the need to reach beyond the effect that procedural and conceptual knowledge have on each other into an exploration of how the effect occurs. While the evidence on the iterative nature of the two forms of knowledge is compelling in these studies of children, it would be useful to determine the nature of their interaction in other populations.

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While I previously stated that Hiebert and Lefevre (1986) did not define conceptual knowledge as verbal knowledge, they did argue that for procedural knowledge to include conceptual knowledge it must contain within it, for the learner, a connection to the real world. In this regard, the operationalization of conceptual knowledge in these three studies (Rittle-Johnson & Alibali, 1999; McNeil et al., 2012; Canobi, 2009) was a complete departure from Hiebert and Lefevre's conceptual and procedural knowledge framework. In all three studies conceptual knowledge was measured as an understanding of the mathematical concepts represented by procedural work. From this vantage point the three studies are were fact completely situated in the procedural knowledge realm, and may in fact be studies of deep procedural knowledge (Star, 2005).

Discussion

In my reading of the literature in preparation for and creation of this paper I could not help but wonder about the validity of splitting mathematics knowledge into procedural and conceptual parts. Though I know that there are other frameworks of mathematical knowledge, the conceptual knowledge and procedural knowledge framework and the terms that accompany it are a lingua franca among mathematics education researchers. While the model of an iterative interplay between the types of knowledge is attractive and demonstrable it seems to me that both forms of knowledge must exist simultaneously within a learner and that experiences that build conceptual understanding may *simultaneously* work to build procedural fluency. Of course if the process is simultaneous, then the experience can hardly be defined as building conceptual or procedural understanding but in truth is simply building understanding.

One idea that is becoming more apparent to me the deeper that I delve into the literature is that researchers cluster around worldviews. The result is that it is difficult to compare contrasting points of view because the language and the meanings of the words are not consistent across tribes. While it is natural for people to be attracted to those of like minds, the effect is that the as a community it becomes nearly impossible to enrich our understanding. I suppose it makes sense that the oldest form of the framework (Hiebert & Lefevre, 1986) saturates the literature, but the careless application of the concepts that draw from presumed understanding rather than the clearly defined construct devalues the research that is built upon it.

I do understand the argument for using the ability to verbalize mathematics understanding as a demonstration of conceptual understanding. When we measure conceptual knowledge using language based responses or language based prompts, we are able to determine whether or not the learner has the ability to connect their symbolic understanding with the larger world. This connection then is made overt. This alone is not enough to answer whether or not the learner is creating a net of knowledge or has made connections that are not language based. How then do

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Comment [7]: A good question.

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Comment [8]: Good hypothesis

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Comment [9]: ??

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Comment [10]: Yes indeed. There is little communication across the walls. A colleague has done literature reviews to examine how much cross-citation there is across disciplinary boundaries (e.g., cognitive science, educational psychology, mathematics education). The answer is not much.

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Comment [11]: Yes - this is why I'm a proponent of reading original works whenever possible. Lots of people cite the person that cited Hiebert & Lefevre rather than the original.... This contributes to dilution.

we find evidence of conceptual understanding in the connections made in the purely symbolic realm of mathematics?

Neuroscience research suggests that high-level math expertise and basic number sense share common roots in a nonlinguistic brain circuit (Amalric & Dehaene, 2016). When considering his inventive processes, Einstein stated:

> "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined." (1945).

It would be absurd to think that the absence of verbal or spoken communication indicates a lack of conceptual understanding in Einstein's processes. This line of reasoning takes me beyond the scope of this paper, and unfortunately, perhaps, outside of the tribe of mathematics education research.

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Comment [12]: This becomes an even greater question the higher in mathematics and the more symbolic mathematics becomes.

References

Amalric, M., & Dehaene, S. (2016). Origins of the brain networks for advanced mathematics in expert mathematicians. *Proceedings of the National Academy of Sciences of the United States of America, 113*(8), 4909-4917. doi: 10.1073/pnas.1603205113

Baloğlu, M. (2002). Construct and concurrent validity and internal consistency, split- half, and parallel-model reliability of the revised mathematics rating scale. *Dissertation Abstracts International, 63,* 6-B. (UMI No. 3058162)

Canobi, K. H. (2009). Concept–procedure interactions in children's addition and subtraction. *Journal of Experimental Child Psychology, 102*(2), 131-149.

Cheng-Yao, L. (2010). Web-Based Instruction on Preservice Teachers Knowledge of Fraction Operations. *School Science & Mathematics*, *110*(2), 59-70.

Cramer, K., Post, T. R., & del Mas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education, 33,* 111-144.

Einstein, A. (1945). Appendix II: A testimonial from Professor Einstein. In J. Hadamard (Author) *The Psychology of Invention in the Mathematical Field*. Princeton, NJ: Princeton University Press)

Hiebert, J., Gallimore, R., Bogard Givvin, K., Hollingsworth, H., Miu-Ying Chui, A., Wearne, D., ... Stigler, J. (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study* (No. NCES 2003–013 Revised). Washington D.C.: U.S. Department of Education, National Center for Education Statistics.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Lamb, L., Bishop, J., Philipp, R., Whitacre, I., & Schappelle, B. (2016, November). The relationship between flexibility and student performance on open number sentences with integers. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.) *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 171-178).

Lin, C., Becker, J., Byun, M., Yang, D., & Huang, T. (2013). Preservice Teachers' Conceptual and Procedural Knowledge of Fraction Operations: A Comparative Study of the United States and Taiwan. *School Science & Mathematics*, *113*(1), 41-51.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Maciejewski, W., & Star, J. R. (2016). Developing flexible procedural knowledge in undergraduate calculus. *Research in Mathematics Education*, *18*(3), 299-316.

McNeil, N. M., Chesney, D. L., Matthews, P. G., Fyfe, E. R., Peterson, L. A. Dunwiddle, A. E., & Wheeler, M. C. (2012). It pays to be organized: Organizing arithmetic practice around equivalent values facilitates understanding of math equivalence. *Journal of educational psychology, 104* (4), 1109 - 1121.

Rayner, V., Pitsolantis, N., & Osana, H. (2009). Mathematics anxiety in preservice teachers- Its relationship to their conceptual and procedural knowledge of fractions. *Mathematics Education Research Journal, 21*(3), 60-85.

Renkl, A., (1997). Learning from worked-out examples: A study on individual differences. *Cognitive Science, 12*, 1-29.

Rittle-Johnson, B, & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 9*(1), 175-189.

Rittle-Johnson, B., Schneider, M., & Star, J. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. Educational Psychology Review, 27(4), 587–597. https://doi.org/10.1007/s10648-015-9302-x

Schoenfeld, A. H. (2004). The math wars. *Education Policy, 18*(1), 253-286

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*, 404–411.

Star, J. R., and Gabriel J. Stylianides (2013). Procedural and Conceptual Knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics, and Technology Education 13*(2), 169-181.

Dr. H's additional comments:

One thing to consider in thinking about the different tribes (and it is a tribal phenomenon) is the predilection toward different methods of gathering evidence and the relationship to how knowledge is defined both in terms of mathematics and in terms of what it means to grow knowledge in the field. In both cases, a design decision is made to privilege certain kinds of knowledge. In some cases, the measures end up defining the knowledge they are measuring and are a boundary on what knowledge can be captured (or not). The other question to consider for

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pretest/posttest studies is the timepoint and when are they needed to capture before/after vs. studies of process where pre/post is not necessarily appropriate.