Designing a Mobile Mathematics Application for Prospective Elementary School Teachers

As undergraduate students who intend to become elementary teachers prepare to enter their teacher training programs the literature indicates that the strength and the nature of their understanding of mathematics is unequal to the task that lies before them. This paper will discuss the state of mathematics knowledge of prospective preservice elementary teachers, students who have completed their regular undergraduate mathematics coursework but have not yet begun their professional training courses. If prospective preservice elementary teachers lack a strong foundation in fundamental mathematics, then they may not be able to develop the pedagogical knowledge that they will need to be effective mathematics educators. There are serious implications of this condition on their professional performance as teachers, possibly negatively impacting the potential mathematics achievement of their future students.

Enlisting the affordances of mobile technologies to help close the gap between what these students know and what they need to know seems a natural fit. Mobile devices are ubiquitous in this population and their always-on, always-handy, internet-connected condition gives them the potential to be a powerful tool. What is currently lacking is a mobile app that is designed to build the foundational mathematics knowledge that this population needs. The nature of that need and the current state of research into the uses and effectiveness of mobile phone applications will be discussed, and the implications for design and development will be explored.

Mathematics Content Knowledge

It seems obvious that teachers must be competent practitioners of mathematics if they are to have any chance of being effective teachers of mathematics to children. Yet research indicates that the level of knowledge of mathematics in U.S. teachers is weak overall (Ball, 1990; Hembree, 1990). The effect of this pervasive weakness is teachers whose understanding lacks coherence (Ma, 1999). A review of research on teacher knowledge and teachers' use of mathematical knowledge (Ball et al., 2001) found pervasive weaknesses in U.S. teachers' understanding of mathematical ideas and relationships. Determining what constitutes the mathematics knowledge necessary for teaching is a complex process complicated by its multidimensional nature, making it difficult to pinpoint what skills and knowledge are present in effective teachers (Hill, Ball, and Schilling, 2008).

To assess mathematics content knowledge it is necessary to define what mathematics content knowledge is and what specializations are desirable for prospective teachers. It is defined in many ways in the literature, both in terms of the nature of the mathematics understood by the learner and in how mathematics knowledge is measured to determine the adequate levels of understanding necessary to be an effective teacher. Mathematics knowledge was split into procedural knowledge and conceptual knowledge in Hiebert and Lefevre's (1986) landmark article that has been used and is still in use in mathematics education research to define mathematics knowledge types. More recent research points to a integration of procedural knowledge and conceptual knowledge with indications that each type when parsed contributes to the strength of the other (Star & Newton, 2009). This general understanding of mathematics knowledge is further refined when looking at the mathematics knowledge that is recognized as crucial to the for effective mathematics instruction. Before they become teachers, they take part in teacher training. These students come to teacher training from their secondary and undergraduate mathematics courses.

Teacher Knowledge

It is unsurprising that the mathematical knowledge that an undergraduate prospective elementary teacher must acquire to adequately teach is more specialized than the mathematical knowledge of the general public. Mathematical knowledge for teaching falls into two categories: content knowledge (CK) and pedagogical content knowledge (PCK). These aspects of teacher knowledge were defined by Shulman (1986) in his landmark article that separated the two types of knowledge in a way that educational researchers still use to investigate teacher content knowledge. Shulman defined content knowledge as "going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter" (pg. 9). Content knowledge includes knowledge of the facts and procedures of mathematics and the understanding of the purpose and relationships within mathematics and beyond mathematics, containing both theory and practice. PCK is the specific forms of content knowledge related to how mathematics is taught. PCK includes alternate ways in which CK can be represented to make it understandable to students, knowledge of the preconceptions and misconceptions that students bring to the subject, and strategies for organizing student understanding. Shulman identified curricular knowledge as a third type of teacher knowledge that is beyond the scope of this paper.

Research indicates that there is a historical and ongoing weakness in preservice teacher mathematics knowledge. A longitudinal study of CK (Ball, 1990) of 252 preservice teacher candidates at the point in which they entered formal teacher education found that, instead of the connected mathematics knowledge described by Shulman (1989), many emerged from their content coursework with skills that were limited to discrete procedural processes disassociated from larger mathematical concepts. Many of the prospective teachers focused on procedures and rules because they perceived the process of doing mathematics as simply following set procedures step-by-step to generate answers: that mathematics itself was an arbitrary collection of facts and rules to be remembered and employed. The author argued that the data suggested "that the mathematical understandings that prospective teachers bring are inadequate for teaching mathematics for understanding," (Ball, 1990, p. 464). A decade later, in spite of curriculum reforms and robust professional development programs, weakness in mathematics content knowledge had not improved. A review of research on teacher knowledge and teachers' use of mathematical knowledge (Ball, Lubienski, & Mewborn, 2001) of studies that looked closely at teacher' knowledge of multiplication, division, rational numbers, functions, geometry, and proofs found pervasive weaknesses in U.S. teachers' understanding of mathematical ideas and relationships.

In the early 2000s an effort was begun to develop measures that could empirically test mathematical content knowledge that teachers possess (Hill, Ball, & Schilling, 2004). The researchers sought to develop a construct that represented mathematical knowledge for teaching (MKT). They pilot tested numerous multiple-choice items intended to represent the mathematical knowledge used in teaching elementary mathematics with 1,552 elementary school teachers. Exploratory factor analysis found three underlying dimensions of MKT: knowledge of content in number concepts and operations, knowledge of content in patterns, functions, and algebra, and knowledge of students and content in number concepts and operations. In addition, they found that specialized knowledge included knowledge of specialized tasks specific to teaching and knowledge of students. They concluded that MKT consists of more than the knowledge of mathematics held by a well-educated adult; there was evidence of more mathematical depth to teaching elementary school. They found that MKT was a multiple-dimensional construct that consisted of a strong knowledge of basic mathematics that provided a foundation for the specialized knowledge that teachers use to teach. The authors state that the additional knowledge may include teacher understanding of why mathematical statements are true and knowledge of multiple representations of mathematical ideas.

As part of the ongoing development of measures to test MKT, Hill et al. (2008) developed a framework that refined Shulman's (1989) division of teacher knowledge into subdomains (Figure 1). It is important to note that the measure of MKT incorporates both subject matter with no knowledge of students or teaching entailed, and Shulman's proposed PCK. The three portions of the oval under Subject Matter Knowledge include mathematics knowledge that does not include knowledge of students or teaching. Common content knowledge (CCK) is mathematics to be taught that used in professions or occupations. Specialized content knowledge (SCK) is the knowledge of how to represent mathematical ideas, and explanations or common rules and procedures. Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics (Zazkis & Mamolo, 2011). The three portions under the right side of the oval under Pedagogical Content Knowledge include knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum.

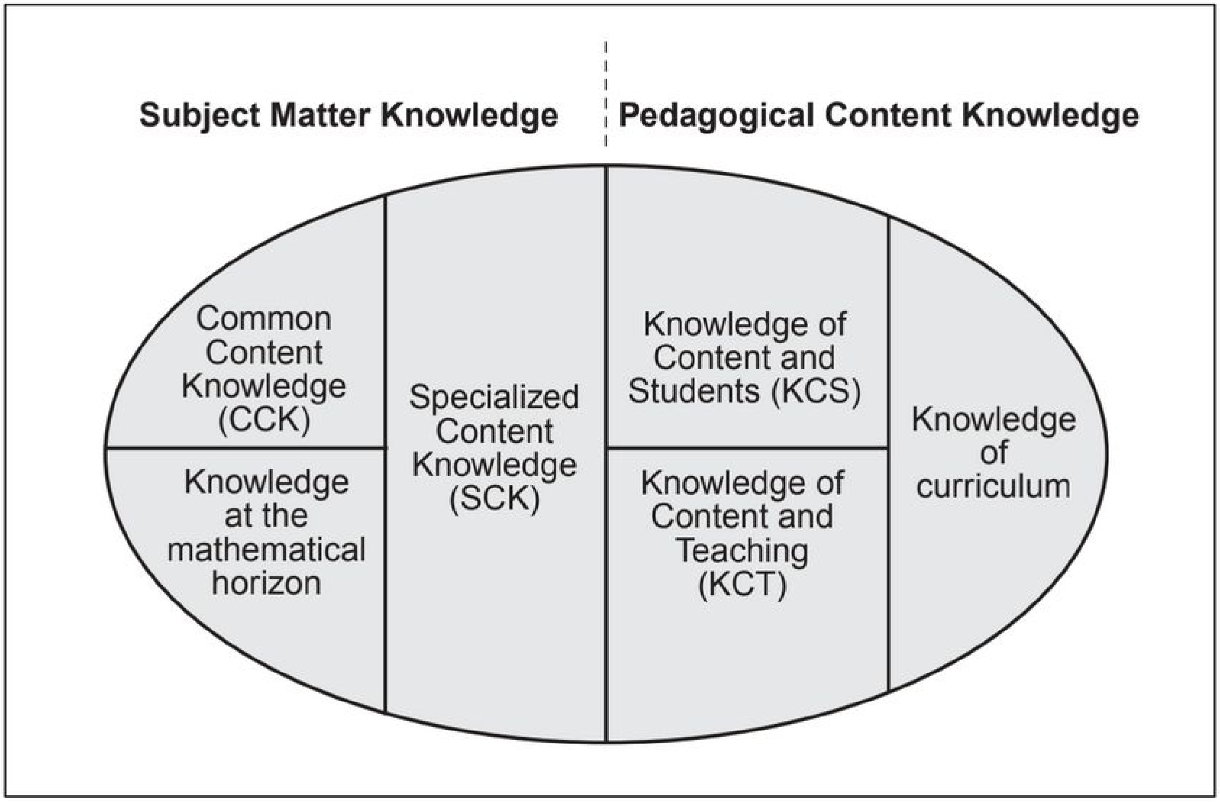

Figure 1. Domain map for mathematical knowledge for teaching (Hill et al., 2008).

A study (Hill et al., 2008) focused specifically on the development of a measure of knowledge of content and students (KCS), a subset of PCK which is a subset of MKT, was conducted using a pretest/posttest design of 640 teachers attending a number and operations professional development course for elementary teachers. This was followed by interviews of 26 K-6 teachers from three Midwestern school districts who were selected on the basis of either low or high CK scores. The researchers found that teachers relied both on familiarly with student error and on mathematical analysis to answer questions correctly. The multidimensionality demonstrated led to problems in measurement of KCS because researchers found that mathematical reasoning and knowledge could compensate for a lack of KCS leading to false positives on the measurement of KCS. Differentiation of the multidimensional aspects of the mathematical knowledge for teaching remained elusive.

Though differentiated understanding of teacher knowledge was obscured by high levels of mathematical reasoning and knowledge in some participants, the participants' ability to compensate for an absence of knowledge of students potentially may shed light on what foundational knowledge effective mathematics teachers possess. In the years since, Shulman (1989) defined the concept of mathematical knowledge that is specific to teaching, a large body of work investigating mathematics teacher knowledge has accumulated. The Hill, Ball, and Schilling (2008) subdivisions of Shulman's SMK and PCK have become the most commonly used measures of teachers' mathematical knowledge (Blömeke & Delaney, 2012). A meta-analysis of 60 research articles that investigated PCK (Depaepe, Verschaffel, & Kelchtermans, 2013) found that all of the studies connected content knowledge and pedagogy. The authors found that conceptualizations of PCK fell into two distinct categories. The first category approached PCK from a cognitive perspective that had provided empirical evidence for a positive connection between teachers' PCK and student learning outcomes. The second category of studies approached PCK from a situated perspective that provided insight into what actually happens in classrooms. In all studies PCK was seen as a form of practical knowledge and content knowledge was described as an important and necessary prerequisite. This finding is a key indicator that strengthening content knowledge in prospective preservice teachers before they begin their teaching methods courses has the potential to support their successful development as effective teachers of mathematics.

Effects on Student Achievement

Measuring teacher content knowledge would be meaningless without an understanding of how MKT may impact student learning. A study of 115 elementary schools **(**Hill, Rowan, & Ball, 2005) investigated how teachers' MKT contributes to students' mathematics achievement. Data included student assessments administered in the fall and spring of each academic year. Teacher data was gathered with a highly structured self-reported log of the time devoted to mathematics instruction, content covered, a survey questionnaire of educational background, involvement in professional development, and language arts and mathematics teaching. The survey was the source of items included in the content knowledge for teaching mathematics measure. The measure was composed of multiple-choice items representing teaching-specific mathematical skills. The study found that teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement. The authors conclude that content-focused professional development and preservice programs will improve student achievement.

Another study investigated the relationship between secondary teachers' MKT and student learning outcomes (Hatisaru & Erbas, 2017) using the framework developed by Hill et al. (2008) to analyze two mathematics teachers and their ninth-grade students ($n=59$) in a vocational high school. The teachers were selected as representing strong and weak knowledge of functions. Measure of MKT was adapted from items used to measure student's knowledge of functions and teachers' knowledge of functions. Follow-up interviews were conducted to obtain a more detailed picture of knowledge of the function concept. The same instrument was used to test student outcomes at the end of a 5 week functions instructional unit. Classroom observations and follow-up interviews were conducted to discuss overall reaction to the class and planning for the next class. The researchers selected key aspects of the MKT that assessed specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). The authors developed a concept framework for evaluating the knowledge domain and used their framework to conduct an qualitative analysis of the data gathered throughout the study. The results indicated that SCK was a necessary condition for KCS, and both aspects influenced instructional practices and learning outcomes.

A recent study (Tchoshanov et al., 2017) of 90 late elementary teachers (grades 5-9) and their students (*N=6,478),* looked closely at the components of MKT and the relationship with student achievement. Rather than using the construct developed by Hill et al. (2008), the connection between MKT and the impact on student performance was studied using an instrument that assessed different cognitive types of teacher knowledge. Cognitive types were divided into knowledge of facts and procedures, including memorization and basic mathematical facts, rules, and algorithms (T1), knowledge of concepts and connections (T2), and knowledge of models and generalizations, conjecturing, generalizing, proving theorems (T3). The authors considered these types as low, medium, and high level knowledge types respectively. Student performance was measured using an end-of-course exam. A correlation found between teachers' content knowledge and student performance with teacher's overall mastery of content knowledge significantly associated with students attaining higher grades in mathematics classes. T1 and T2 were significantly correlated. T3 was not. Strength in T1 appeared to be necessary for high performance at the T2 level.

In spite of the difficulties of measuring teacher knowledge, across the studies, the consistent findings were that knowledge for teaching mathematics is dependent upon a foundational knowledge of integrated mathematics. Liping Ma's (1999) book explores the disparity between Chinese and American student outcomes, with Chinese students typically outscoring U.S. students on international comparisons of mathematics competency. Ma analyzed the fundamental mathematics that Chinese and American elementary teachers bring to their teaching and found a striking contrast between the nature of the knowledge between teachers in the two countries. U.S. teachers tended to be procedurally focused, competent with whole numbers, but challenged by more advanced topics such as division of fractions and perimeter and area of rectangles. In comparison, she found that Chinese teacher knowledge was "coherent while that of the U.S. teachers was clearly fragmented." (p. 107). Chinese teachers described why algorithms worked and when discussing alternative ways to solve problems explained that alternate approaches were possible, not because of isolated rules, but because of the underlying relationships that connect operations. Conceptual and procedural topics were interwoven and provided the groundwork to build further mathematics understanding. Ma identified this type of mathematical content knowledge as a *profound understanding of fundamental mathematics* (PUFM). Teachers who possess PUFM are able to conceive of the ideas that are connected to the structure of the discipline of mathematics. This is strikingly similar to the T1 and T2 cognitive types (Tchoshanov et al., 2017) When challenged to demonstrate why an algorithm works they used not only verbal explanations and examples, they also justified their explanations with symbolic derivations.

Ma (1999) states that PUFM "is the mathematical substance of elementary mathematics that allows for a coherent understanding of it" (p. 118). Elementary mathematics "means providing them with a groundwork on which to build future mathematics learning." (p. 117).

Teachers cannot provide this groundwork unless they themselves have a coherent understanding of mathematical practices and constructs. Ma also talks about depth of understanding as "connecting it with more conceptually powerful ideas of the subject" (p. 121). If this depth of understanding is present then as teachers they will be able to "reveal and represent them in terms of teaching and learning" (p. 122). Ma defined four properties of this representation:

1. **Connectedness** refers to the connections among mathematical concepts and procedures, prevent students' mathematics knowledge from being fragmented into isolated topics.
2. **Multiple perspectives** an appreciation of various approaches to a solution leading to a flexible understanding.
3. **Basic ideas** of the principles of mathematics and use them to guide students in real mathematical activity
4. **Longitudinal Coherence** is a fundamental understanding of the whole of elementary mathematics curriculum freeing teachers to review concepts previously encountered and strategically prepare students for subsequent learning.

While *basic ideas* form the building blocks of mathematics, the other four properties are in essence refinements of coherent connectedness. If this connected understanding is not present in prospective preservice teachers, in what way can it be cultivated? Before addressing this issue it is necessary to define what *connectedness* means in mathematics teaching and learning.

Theoretic Framework

Procedural and Conceptual Knowledge

When Ma (1999) describes the interweaving of conceptual and procedural knowledge she refers to procedural knowledge as the specific steps used in solving problems and conceptual knowledge as the rationale that supports the procedures and the knowledge of connections to related mathematical concepts. This understanding of the differences between procedural and conceptual knowledge was defined by Hiebert and Lefevre (1986) in their landmark article to provide a useful way for researchers to understand student learning processes. While Ma does not reference this framework, her use of the concepts align: especially when considering the interplay between the two forms. While Hiebert and Lefevre's purpose for developing the framework was to aide mathematics education research, they acknowledged that the relationship between procedural knowledge and conceptual knowledge was not well understood, and that often mathematics knowledge may be an inseparable combination of both forms of knowledge. In spite of this, they argued that distinguishing types of knowledge would provide a way to understand the failure or success of building mathematics understanding.

Hiebert and Lefevre (1986) define conceptual knowledge as a connected web of knowledge: a network of linked relationships that are as important as the discrete elements. The web of knowledge is conceptual only if the learner recognizes the relationships between elements. Conceptual knowledge not only describes the connections between known elements, it also describes the connections made between existing knowledge and newly acquired knowledge. The context is created by the network of relationships, allowing the learner the freedom to apply existing knowledge to novel problems.

Procedural knowledge as described by Hiebert and Lefevre (1986) is the use of the formal language and the symbolic representation system of mathematics. Algorithms or rules are used to complete tasks with an awareness of only surface features. The knowledge of the meaning of the processes is not accessed or necessary for successful completion of tasks. Step-by-step instructions prescribe how to complete tasks. These tasks may then be sequenced into superprocedures that incorporate lower level subprocedures. Procedural knowledge allows students to solve complex superprocedures as a chain of prescriptions without knowledge of the meaning of the task.

Over time, referencing Hiebert and Lefevre's framework (1986) has become a kind of shorthand for the understanding that procedural knowledge essentially amounts to meaningless number crunching, and that conceptual knowledge is application to the real world demonstrated by the conversion of mathematical forms into plain language. There has been both a tacit and a sometimes overt belief that conceptual knowledge defined in this way is a higher form of understanding and a more valid metric to be used when assessing mathematics knowledge than demonstrated procedural skill (Star, 2005). This simplistic application of the framework does not take into account the intersection and influence that one type of knowledge may have on the other. In three studies that used Hiebert and Lefevre's framework to compare procedural knowledge to conceptual knowledge in learners (Cramer, Post, & del Mas, 2002; Rayner, Pitsolantis, & Osana, 2009; Cheng-Yao, 2010) the ability of the participants to solve equations was used only to assess procedural knowledge. Conceptual knowledge was measured by the participant's ability to create word problems that demonstrated an understanding of the concept being investigated. There was no provision to gather data on participants' conceptual understanding that may have been embedded in their procedural work. When conceptual understanding is only tested by asking participants to create scenarios that would embody given mathematical expressions, conceptual theoretical mathematical understanding that may be present but not applicable in a real-world context is not accessed or assessed. The oversimplification of mathematics knowledge into such narrowly defined constructs stifles exploration of the connected mathematics that Ma (1999) observed and the complex interplay that Hiebert and Lefevre acknowledged.

Star (2005) addressed the limitations that the bifurcation of mathematical knowledge into conceptual and procedural types had imposed on mathematics education research and the barriers that the simplistic model created to the development of a meaningful understanding of the complexity of the relationship between the two concepts. He argued that contrary to the common perception of procedural knowledge as superficial and rote and the perception that conceptual knowledge is knowledge that is known deeply, that both types of knowledge can be either superficial or deep or anything in between. He described this depth of knowledge as *knowledge quality.* In discussing deep procedural knowledge he describes environments where procedures are utilized with a knowledge of the justification for use and the understanding of the impact the environment or situation may have on selection of procedure type or method.

These rich relationships echo the connected mathematics of Ma (1999) and the web of knowledge described by Hiebert and Lefevre (1986) as conceptual knowledge. Of course the semantics begin to cause some difficulty, but I believe that Star's (2005) article is, rather than a refutation of Hiebert and Lefevre (1986), a reassertion of the spirit of the framework as it was originally developed. Hiebert and Lefevre do not propose at any point that language embodies a higher form of knowledge or equate language with conceptual knowledge. Looking at procedural knowledge without reference to conceptual acknowledges the importance of procedural knowledge as valuable in and of itself (Star, 2005).

Star (2005) argues that deep procedural knowledge manifests as *procedural flexibility*. In a qualitative study that investigated the development of procedural flexibility by experts, procedural flexibility was defined as the knowledge of multiple procedures relevant to solving a particular task, and the ability to select the most appropriate among these to complete a task (Star & Newton, 2009). The participants included two mathematicians, two mathematics educators, two secondary mathematics teachers, and two engineers. They were given a 55-item test of symbolic mathematics problems that were designed to ensure opportunities to demonstrate flexibility. Semi-structured interviews were conducted following the test that allowed the experts to explain the strategies they had used and their perceptions of alternative strategies that they were aware of. The researchers found that the experts demonstrated knowledge of and use of multiple strategies and generally expressed a preference for the most efficient strategies, those with the least number of operations or least arithmetic complexity, for a given problem. When the experts failed to use the optimal strategy when problems could be solved using well-practiced, automatized approaches. The researchers found that the experts believed that flexibility is best developed implicitly and individually by the repeated problem solving experiences of learners.

To investigate what types of experiences would prompt the development of procedural flexibility, a recent study (Maciejewski & Star, 2016) examined the effects of a teaching intervention designed to promote flexibility in procedural knowledge. The researchers sought to determine not only if procedural flexibility could be developed, but also if it resembled expert-like procedural performance. The design was quasi-experimental pretest/post test. Two sections of an introductory calculus course for first year college students, taught by the same instructor, were selected for the experimental study. A pretest on differentiation was given. After receiving a lesson that contained a traditional sequence of instruction on procedures, the control section was given a typical worksheet for homework. The treatment section was given a worksheet that specified two approaches for solving each assigned problem and were asked to describe which method they prefered. The homework assignment was followed by a readministration of the pretest as a post test. An analysis of the data found that here were no significant differences in the sections' score averages; both groups demonstrated higher achievement on the posttest. The treatment group used a greater variety of strategies than the control group and moved closer to expert-like performance. The authors concluded that it is possible to use an instructional task to support the development of undergraduate students' flexible use of procedures. Because the control group did not demonstrate flexibility after practice, the authors determined that an activity that prompted critical reflection by presenting tasks that prompt students to resolve questions in different ways and allow for the comparison of different solutions may support the development of deep procedural knowledge.

Procedural Flexibility and Performance

The lack of empirical research on procedural flexibility in prospective preservice teachers necessitates looking at the relationship between student flexibility in procedural problem solving and mathematics performance in another population. A qualitative study (Lamb et al., 2016) used clinical interviews to investigate the effect of procedural flexibility in students grades 2, 4, 7, and 11. The researchers sought to determine the degree in which flexible ways of reasoning influenced performance on integer problems. The wide grade span was chosen to cover student learning experiences from those who had not yet received school-based integer instruction to those who were enrolled in precalculus or calculus courses and therefore deemed to be successful high school mathematics students. Individual clinical interviews were conducted and videotaped at the students' school sites. The interviews were standardized and all students were asked to complete the same 25 open number sentences. Interviews were coded both for underlying reasoning and for correctness. Five categories were used to identify ways of reasoning: order-based, analogy-based, computational, formal, and developmental. Flexibility was measured by the variety of methods students used to solve tasks. Proficiency with a particular form of reasoning was demonstrated when a student used it three or more times. The number of ways that students used forms of reasoning that they were proficient in was the measure of flexibility.

Case studies were performed on three 7th grade students who exemplified the relationship between flexibility and accuracy. The first student chosen completed 32% of the open number sentences correctly. The second student completed 64% of the problems correctly. The third student completed 100% of the problems correctly. The researchers had found that the seventh grade students had had the greatest spread in flexibility. Eighty-five percent of 11th graders had used 3 or 4 methods. Flexibility scores for both groups had correlated positively with performance. This result also held across the case studies. The first and second cases used one type of reasoning almost exclusively which limited their options for solving problems and negatively impacted their success. The third case, who had completed every open number sentence correctly, flexibly used a wide range of strategies on the problems and appeared to choose strategies that corresponded with the underlying structure of the open number sentence. The authors concluded that students who rely on a single way of reasoning may be impeded in their success because of their limited flexibility and that multiple ways of reasoning promotes successful performance. For every participant group, the correlation between flexibility and accuracy held; more flexible students were more successful.

Procedural Flexibility and Content Knowledge for Teaching

The sum total of the literature reviewed provides a substantial justification for the development of an intervention that could conceivably strengthen the procedural flexibility of prospective preservice teachers. The presence of procedural flexibility is an indicator of strategic thinking in mathematics problem solving (Lamb et al., 2016), and appears to have been strengthened when students were presented with multiple ways to approach problems (Maciejewski & Star, 2016). Star (2005) identifies procedural flexibility as an exemplar of deep procedural knowledge that "is associated with comprehension, flexibility, and critical judgement and that is distinct from (but possibly related to) knowledge of concepts" (p. 408). An investigation of the procedural flexibility of experts (Star & Newton, 2009) found that they solved problems with a combination of automized skill and a thorough understanding of the underlying concepts involved. This combination of skill and the ability to connect with more powerful concepts, core to Ma's (1999) PUFM, was present in effective teachers (Tchoshanov et al., 2017), and was identified as a necessary prerequisite to the development of pedagogical skill (Depaepe et al., 2013).

Technology Based Learning Interventions

If the primary strength of undergraduate mathematics knowledge is in procedural knowledge, then it makes sense to use students' existing procedural knowledge as a tool to build conceptual understanding. Rather than perceiving the shallow procedural knowledge of undergraduate students as a liability, perhaps their existing knowledge can function as the starting point, the building blocks, that can be used to create deep procedural knowledge by providing the opportunity for students to build the net of knowledge that interconnects the disparate parts of their existing mathematics understanding. Mobile technology platforms are a natural environment to provide the constant practice that expert mathematicians believed was necessary to their skill acquisition (Star & Newton, 2009). And software for mobile devices can be designed to deliver the type sequenced instruction that prompted procedural flexibility (Maciejewski & Star, 2016).

It is a logical choice employ the use of mobile technologies, specifically mobile phones, to support the mathematical content learning of prospective preservice elementary teachers. This generation of students is highly likely to have the technology with them at all times, and they have shown an affinity for using mobile phones for learning on their own (Chen, Seilhamer, Bennett & Bauer, 2015). We expect our elementary school teachers to be proficient in using technology with their students (NCTM, 2014; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), even though it is unlikely that they have had experience using technology for learning in their own coursework (Chen, Seilhamer, Bennett & Bauer, 2015). Increasingly, technology use means mobile devices (Johnson, Adams, Becker, Estrada, Freeman, & Hall, 2016). Few university instructors design exercises that use mobile technology, and more than half bar mobile devices from the classroom (Dahlstrom &

Bichsel, 2014). There is evidence that this generation of students, who are often described as "digital natives," struggle to successfully integrate mobile technologies into their classrooms once they enter practice, especially when teaching mathematics (Orlando & Attard, 2016). While there are sure to be many factors that contribute to the struggle, it is likely that lack of experience using mobile technologies in their own coursework is a contributing factor.

A 2014 university-wide survey (Chen et al., 2015) of undergraduate (n=1,075) and graduate (n=106) students found that 95 percent owned a smartphone device. Students used the devices to look up lecture topics, discipline-specific apps, and course textbooks. Students perceptions of the benefits of using mobile devices included increased knowledge in their fields of study, increase quality of work, and greater motivation complete coursework. Sixty-six percent of students reported using a mobile app for learning on their own at least once a week. More than half of the students indicated that they would prefer that instructors not use mobile apps because of lack of technical support and training. There is little research on the effectiveness of learning with mobile phones, especially in this population. This may be due to the relative recent near universal adoption of these device by university students, the lack of sufficient instructor training (Chen et al., 2015), or the poor quality of existing mathematical apps (Larkin, 2015).

Assessing the quality of mathematics educational apps

Educational mathematics apps are heavily represented in both the App Store (Apple) and the Google Play Store (Android): the two largest marketplaces for apps and the most likely sources for undergraduates who use apps. A review of the mathematics apps available reveals myriad intensely colored child-like games-based apps and sombre looking apps designed for calculus practice. There appear to be few apps designed for adults to use to strength fundamental mathematics skills. Of courses, how an app appears in an app store does not given an indication of the software's suitability or quality. There are ongoing efforts to create a means to assess the usefulness of apps for research and for use (Lee & Cherner, 2015), but a framework or other construct for analysis has not yet emerged that is in broad use.

There are indications that there is no consensus among preservice elementary teachers of what constitutes a useful mathematics educational app. A study (Handal, Campbell, Cavanagh, & Petocz, 2016) of 373 elementary education students from three universities sought to establish the construct validity of an instrument designed to assess the perceived usefulness of educational apps in mathematics education by preservice teachers. Participants were asked to examine an app on an iPad that had been determined by the researchers to be pedagogically and technically sound and record their responses on the developed instrument. Quantitative analysis of the responses found that construct validity could not be established because responses had no discernible patterns. The authors argued that this result was most likely due to students' lack of understanding of the appropriate role that technology should play in learning. I think that this is not only a reflection of a lack of understanding how technology will work in the lives of their future students, but also a reflection of a lack of understanding of how technology may enrich their own learning experiences.

Affordances of Mobile Technology

The ubiquitous presence of mobile phones in students' lives is in itself a compelling reason to develop an app based intervention to support the mathematics content learning of prospective preservice elementary teachers. Students are not likely to use a resource that is not easily accessible. In addition to the ubiquitous nature of the devices, digital technology generally and mobile devices specifically provide multiple affordances that can be leveraged in the development of an intervention. The lack of a mobile app that is designed to develop an integrated concept of mathematics topics that will support procedural flexibility in undergraduate students presents an opportunity to fill the void.

In mathematics education there is a history of developing software that is designed to bring the complex concepts of advanced mathematics to users who would normally not engage in mathematics at that level. In 1994, Kaput and his team began the development of SimCalc: Software for personal computers that was designed to bring the calculus concept of rates of change and advanced algebra concepts to children (Hegedus & Roschelle, 2013). The project would continue for more than 20 years and receive NSF funding that exceeded 16 million dollars (National Science Foundation, 1993, 1997, 2000a, 2000b, 2002, 2004a, 2004b). Kaput wrote that "SimCalc is a technology and curriculum research and development project intended to democratize access to the basic ideas underlying calculus beginning in the early grades and extending to AP calculus and beyond," (1999, p. 155). While the software is primarily known in the mathematics educational research community for its multiple representations of concepts of change, I am more interested in the aspect of the project related to the democratization of advanced mathematics: the use of the affordances of technology to design an environment that invited students to explore mathematical constructs that were considered beyond their grasp.

Ma (1999) found that the Chinese teachers possessed a connected understanding of mathematics that informed their pedagogical approaches. This connected understanding of mathematics is reflected in Kaput's belief that algebra could become an engine of mathematical power for students who were struggling if they were introduced to the concepts as a web of knowledge and skill. He argued that if students were introduced early to the practice of forming patterns and generalizing quantitative reasoning that they then would build deep and varied connections with all of mathematics (Kaput, 1998). The affordances of technology can be used as a tool to expand students' prior knowledge to represent both familiar and unfamiliar concepts in close approximation and to sequence concepts with increasing complexity (Kaput, 2009; Roschelle & Kaput, 1996). An additional affordance that technology environments can provide is the personalization of the users experience. Clinical interviews (Kaput, 1999) with users early in development found that they took ownership of the computer environment when they were given the ability to personalize the experience by means as simple as changing colors and assigning names.

College students expressed aversion to using software in coursework in part because of a lack of available effective training and support (Chen et al., 2015). Application software for mobile phones should be designed to be accessed anywhere at any time. The challenge then is to design software that provides training and support for the user as part of its design. Though SimCalc was developed as a tool to be used in a classroom setting with an instructor, research on its effectiveness provided some intriguing results that may offer insight into dealing with this challenge. Throughout its development, SimCalc was designed to provide onramps to complex mathematical concepts to students at all levels of mathematics development from early elementary through high school. Researchers found (Hegedus & Kaput, 2003) that *all* students performed better using the software relative to their prior knowledge. As the research on SimCalc was scaling up (National Science Foundation, 2004a), a pilot study (Tatar et al., 2008) of the use of SimCalc in the classroom was conducted with 21 seventh-grade mathematics teachers and their students using a pretest-posttest randomized experiment design. Participants came from multiple districts with widely varying socio-economic conditions. Control teachers received no training on the software. Though both teachers and students in the treatment condition had gains were higher than those in the control group, all participants had significant gains in mathematics knowledge. This held across variations in teacher knowledge, experience, and teaching contexts. Of course, there are enormous limitations to generalizing the use of technology that is classroom-based to an intervention that will be used without the simultaneous support of a classroom or an instructor. Yet, it is reasonable to assume that connected mathematics that underlies the design of the software is the constant behind the positive results of these studies. The advancements in mobile technology in the intervening years presents an irresistible opportunity to advance this work into the context that undergraduate students inhabit.

Embodied cognition and mathematics

Swipe movements are now second nature when interacting with mobile phones. This tactile interaction with the device has interesting implications for mathematics learning. Long before the advent of the smartphone, Kaput (1999) identified the potential power that movement and gesture in digital environments could have on learners. He argued that images that responded to users movements would deepen their understanding of the underlying mathematics. Later (Kaput, 2009), as he was exploring the use of SimCalc on handheld devices, he argued that gesture could be used to connect students to purely symbolic representations of mathematics. Though there were glimpses of the power that digital environments could have when users displayed ownership of their online work, Kaput could not have imagined the strength of the immersive effect that our mobile devices can now have on us, subsuming us into digital realities that have the immediacy and salience of the physical world around us. In the early 20th Century Husserl (2005) anticipated this phenomenon as the concept of pure possibility: the experience of imagined objects having the same effect on us as concrete objects. His ideas were later supported by a landmark neuroscience study (Gallese & Lakoff, 2005) that found evidence that conceptual knowledge is mapped within our sensory-motor system, and that this characterizes the way that we function in the world. The same structure that moves us and gives us structural perceptions also structures abstract thought. This form of embodied perception is described as embodied cognition.

Mathematical imagination is bound by the logical necessity inherent in mathematics and therefore does not have the complete conceptual freedom that pure possibility allows and, as Kaput (2009) had conjectured, movement can connect the figural symbolic entities of mathematics to the physical world. A study (Nemirovsky & Ferrara, 2009) of 21 high school students used body motion with technology to integrate mathematics learning to explore the possibility that mathematical propositions can reside in the body. The teaching experiment was designed for the learning of trigonometry concepts involving body motion. Students used a laser pointer to trace angles as they discussed types of triangles. In the follow-up discussion students displayed complex ideas both verbally and with hand gestures to illustrate their understanding. A further evaluation (Hutto, Kirchhoff, & Abrahamson, 2015) of an earlier study (Reinholz, Trninic, Howison & Abrahamson, 2010) that compared the conceptual understanding of 20 grade 4-6 students who were trained to move their arms and hands in a way that would enact proportional relationships. The treatment group significantly outperformed the control group on measures of conceptual understanding of proportional reasoning. The question of whether embodied cognition extends to the affordance of gesture control that tablets provide was explored in the a study of 61 8-11 year olds (Agostinho et al., 2015). The control group was instructed to look at information that was highlighted and circled on an iPad app. The treatment group was instructed to trace the same information on an iPad app using their index finger. The treatment group achieved a higher performance result on a test of recall than participants who studied the material without tracing. The authors conclude that gesturing has a positive impact on fundamental educationally relevant cognitive functions.

Research

There is every indication that swipe gestures have the potential to support mathematics understanding in users, but there remains a need for further research into what role embodied cognition plays in learning on the small screens of smartphones in undergraduate students. Frankly there is a need for research into all aspects of mobile technology and learning. Though the use of mobile devices has grown rapidly, reaching near saturation levels in undergraduates, research into the impact of mobile devices on learning has not kept pace. At this early stage of the development of research into the use and effect of mobile devices, focus is mostly on the potential of mobile devices for teaching and learning mathematics, affective studies on the use of mobile devices, and the use of mobile devices in mathematics teacher education (Borba et al., 2017; Larkin & Calder, 2016). I found in the current published studies on mobile devices and teacher education a proposal to use smartphones to record videos and share understanding in teacher education sessions (Yerushalmy & Botzer, 2011), and proposals for ways to increase prospective teacher engagement in learning with mobile devices (Holden, 2016; Schuck, 2016);

in line with Bourba's (2017) findings that research is currently focused on potential uses rather than empirical outcomes.

Implications for Design and Development

The fact that there is so little prior empirical research into the effectiveness of mobile phone applications in mathematics learning necessitates the need to look at related research on learning and technology. While Kaput functioned in a vastly different world, and designed for devices bound in place with use mediated by instructors, his successful approaches to how and what students can learn in a digital environment provide a solid place to begin in designing an intervention. In spite of the relatively small screen size of mobile phones, the nature of the interaction and the quality of the visuals are exponentially more immediate than those that were available to Kaput on personal computers. Accessibility to technology is not the concern that it was when SimCalc was being developed, but there are new issues that must be addressed. Compatibility across the major platforms (Apple, Android, and web-based design), security of user data, availability of development resources are just the most obvious challenges among others that are sure to surface as development progresses. Design based research methodology allows a way forward by providing a framework for both design and empirical research throughout the development of an intervention. Interestingly the decades-long process that Kaput used, reflected in the NSF Grants, fits the frameworks for design based research currently in use.

Design Based Research

The presence of a learning application on a users mobile device has a potential impact on the users' perception of the device, user experience, the content, and the environment. Within this complex setting, design based research provides a model for rigorous cycles of applied research to "effect change in a learning context through the building of a design intervention through which we uncover pedagogical principles that may be applicable and researchable in similar situations" (Bannan, Cook, & Pachler, 2015, p. 3). As a qualitative method, design based research allows for the uncovering of analytic generalizations from the specifics of the phenomenon being investigated.

Figure 2 (as cited in McKenney & Reeves, 2012, p. 16) illustrates the recursive nature of the design-based research process. Each loop represents a cycle of research with each cycle informing the direction and design of each succeeding cycle.

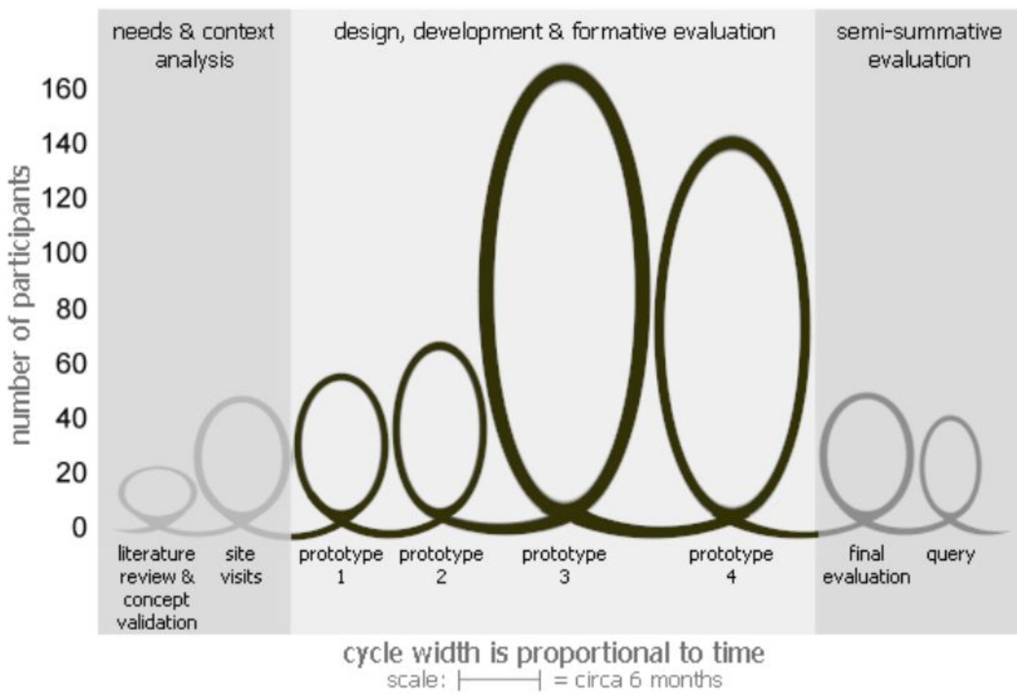

Figure 2. (McKenney & Reeves, 2012, p. 16)

The McKenney and Reeves (2012) framework describes the workflow of a specific research study and provides a clear organizing framework for the progression of the development of a design based research study. While the looping form illustrates individual recursive design/research cycles, the direction of progress is continuously forward. Bannan's ILD Framework (Figure 3) shares the conceptual organization of the McKenney & Reeves with the Informed Exploration phase covering the needs and content analysis; the Enactment phase incorporating design, development, and formative evaluation; and the Enactment: Local Impact phase aligning with semi-summative evaluation. Bannan's Evaluation: Broader Impact phase moves beyond the project development process covered by the McKenney and Reeves framework and propels the project into real-world. In contrast to the McKenney and Reeves framework, the ILD Framework provides multiple avenues of exploration and research that will allow the trajectory of the project to change as more knowledge and experience is gained through the recursive research process.

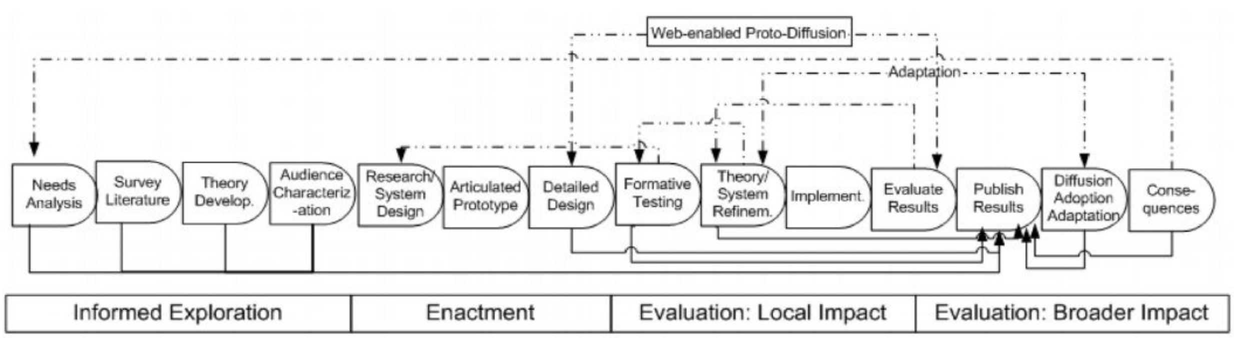

Figure 3. ILD Framework (Bannan-Ritland, 2003, p. 22)

Using the McKenney and Reeves model to sequence workflow will allow for forward movement in development while simultaneously conducting research at every iteration. The integrative learning design (ILD) Framework for mobile learning development (Bannan et al.,

2015) presents guiding questions and applicable research methods for each of the phases of development and allows the design and research cycles to determine the trajectory of the research. Using this construct, the McKenney and Reeves framework can guide the sequence of the research process, and the ILDF can define the theoretical concepts being investigated and the choice of research methodologies that will be used within each cycle of research. Because of the qualitative nature of design based research and the way in which prior cycles influence the trajectory of subsequent cycles, the research design has the flexibility to respond to the expected and the unforeseen challenges on developing an mobile application.

Reflection

No matter how well topics and contents are sequenced, it would be foolhardy to imagine that the complexity of learning mathematics can be reduced to the capabilities of a single mobile application. Still, leveraging the affordances that technology provides to sequence and present mass amounts of information and skill practice presents the possibility of developing a powerful tool to increase mathematical knowledge and understanding. In spite of the fact that mobile devices have become ubiquitous in our world, research on mobile technology's impact on prospective preservice teachers is in its infancy. There is a need for greater exploration of the impact of technologies that allow user autonomy in choosing where, when, and what learning will occur. Though these broad questions certainly cannot be answered in the development of a single mobile application, conducting this study will contribute to the the body of knowledge in this domain.

References

Agostinho, S. S., Tindall-Ford, S., Ginns, P., Howard, S., Leahy, W., & Paas, F. (2015). Giving learning a helping hand: Finger tracing of temperature graphs on an iPad. *Educational Psychology Review, 27*(3), 427-443.

Ball, D. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449-466.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching (4th ed.).* New York: Macmillan.

Bannan, B., Cook, J., & Pachler, N. (2015). Reconceptualizing design research in the age of mobile learning. *Interactive Learning Environments, 24*(5), 938-953

Blömeke, S., & Delaney, S. (2012). Assessment of teacher knowledge across countries: A review of the state of research. *ZDM, 44*(3), 223–247.

Borba M.C., Askar P., Engelbrecht J., Gadanidis G., Llinares S., & Aguilar M.S. (2017). Digital Technology in Mathematics Education: Research over the Last Decade. In: Kaiser G. (ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp 221-233). ICME-13 Monographs. Cham, Switzerland: Springer Nature.

Chen, B., Seilhamer, R., Bennett, L., & Bauer, S. (2015). Students' mobile learning practice in higher education: A multi-year study. *Educause Review.* Retrieved from http://er.educause.edu/articles/2015/6/students-mobile-learning-practices-in-higher-education-a-multiyear-study

Cheng-Yao, L. (2010). Web-Based Instruction on Preservice Teachers Knowledge of Fraction Operations. *School Science & Mathematics*, *110*(2), 59-70.

Cramer, K., Post, T. R., & del Mas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education, 33*, 111-144.

Dahlstrom, E, & Bichsel, J. (2014). ECAR study of undergraduate students and information technology. (Research Report) Retrieved from Educause website: http://www.educause.edu/ecar

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34*, 12-25.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 22*(3–4), 455–479. Doi: 10.1080/02643290442000310.

Handal, B., Campbell, C., Cavanagh, M., & Petocz, P. (2016). Characterising the perceived value of mathematics educational apps in preservice teachers. *Mathematics Education Research Journal, 28*(1), 199-221.

Hatisaru, V., & Erbas, A. K. (2017). Mathematical knowledge for teaching the function concept and student learning outcomes. *International Journal of Science and Mathematics Education, 15*(4), 703-722.

Hegedus, S. & Kaput, J. (2003). *The effect of SimCalc connected classrooms on students' algebraic thinking.* In N. Pateman, B. Dougherty, and J. Zillox (eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th meeting of PME-NA,* Honolulu, HI, College of Education, University of Hawaii, 3, pp. 47-54.

Hegedus, S., & Roschelle, J. (2013). *The SimCalc Vision and Contributions. Advances in Mathematics Education.* Dordrecht, Netherlands: Springer.

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal of Research in Mathematics Education, 21*(1), 33-46.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Hill, H.C., Ball, D.L., & Schilling, S.G. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105*(1), 11-30.

Hill, H. C., Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Holden, J. I. (2016). Mobile inquiry-as-play in mathematics teacher education. *On the Horizon; Bradford, 24*(1), 71-81.

Husserl, E. (2005). *Pure possibility and phantasy. Phantasy, Image Consciousness, and Memory Volume 11 of the series Edmund Husserl Collected Works*. Netherlands: Springer.

Hutto, D. D., Kirchhoff, M. k., & Abrahamson, D. D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. *Educational Psychology Review, 27*(3), 371-389.

Johnson, L., Adams Becker, S., Cummins, M., Estrada, V., Freeman, A., and Hall, C. (2016). *NMC Horizon Report: 2016 Higher Education Edition*. Austin, Texas: The New Media Consortium.

Kaput, J. J. (1998). Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power by "Algebrafying" the K-12 Curriculum. *The nature and role of algebra in the K-12 Curriculum: Proceedings of a National Symposium*, (pps. 25-26). National Research Council, National Council of Teachers of Mathematics and Mathematical Sciences Education Board, Center for Science, Mathematics, and Engineering Education. Washington D. C.: National Academies Press

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Rombert (Eds.), *Mathematics Classrooms that Promote Understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (2009). Building Intellectual Infrastructure to Expose and Understand Ever-Increasing Complexity. *Educational Studies in Mathematics, 70*(2), 211-215.

Lamb, L., Bishop, J., Philipp, R., Whitacre, I., & Schappelle, B. (2016, November). The relationship between flexibility and student performance on open number sentences with integers. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.) *Proceedings of* *the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 171-178).

Larkin, K. (2015) "An App! An App! - My Kingdom for an App": An 18 month quest to determine whether apps support mathematical knowledge building. In T. Lowrie & R. Jorgensen (Eds.), *Digital games and mathematics learning: Potential, promises and pitfallhs* (pp. 251-276). New York City: Springer Press.

Larkin, K., & Calder, N. (2016). Mathematics education and mobile technologies. *Mathematics Education Research Journal, 28*(1), 1-7.

Lee, C-Y. & Cherner, T. S. (2015). A comprehensive evaluation rubric for assessing instructional apps. *Journal of Information Technology Education: Research, 14*, 21-53.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Maciejewski, W., & Star, J. R. (2016). Developing flexible procedural knowledge in undergraduate calculus. *Research in Mathematics Education*, *18*(3), 299-316.

McKenney, S. & Reeves, T. C. (2013). *Conducting educational research design.* New York: Routledge.

National Council of Teachers of Mathematics (2014). *Principles to Action.* Reston, VA: NCTM

National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). *Common Core State Standards*. Washington D. C.: National Governors Association Center for Best Practices, Council of Chief State School Officers

National Science Foundation (1993). *SimCalc: Simulations for calculus learning.* Award #9353507. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=9353507&HistoricalAwards=false

National Science Foundation (1997). *SimCalc: Democratizing access to the mathematics of change.* Award #9619102. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=9619102&HistoricalAwards=false

National Science Foundation (2000a). *Understanding classroom interactions among diverse, connected classroom technologies.* Award #0087771. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=0087771&HistoricalAwards=false

National Science Foundation (2000b). *Planning a rigorous experimental trial of SimCalc's approach to increasing access to complex mathematical ideas.* Award #0089094. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=0089094&HistoricalAwards=false

National Science Foundation (2002). *Scaling up SimCalc: Professional development for integrating technology to teach more complex mathematics (phase 1).* Award #0228515. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=0228515&HistoricalAwards=false

National Science Foundation (2004a). *Working with teachers and leveraging technology to scale opportunities to learn more complex and conceptually difficult middle school* *mathematics.* Award #0437861. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=0437861&HistoricalAwards=false

National Science Foundation (2004b). *Representation, participation and teaching in connected classrooms.* Award #0337710. Retrieved from https://www.nsf.gov/awardsearch/showAward?AWD_ID=0337710&HistoricalAwards=false

Nemirovsky, R., & Ferrera, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*, 159-174.

Orlando, J., & Attard, C. (2016). Digital natives come of age: The reality of today's early career teachers using mobile devices to teach mathematics. *Mathematics Education Research Journal, 28*(1), 107-121.

Rayner, V., Pitsolantis, N., & Osana, H. (2009). Mathematics anxiety in preservice teachers: Its relationship to their conceptual and procedural knowledge of fractions. *Mathematics Education Research Journal, 21*(3), 60-85.

Reinholz, D., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: Embodied artifacts and the guided emergence of mathematical meaning. In P. Brosnan & D. Erchick (Eds.), Proceedings of the thirty-second annual meeting of the north-american chapter of the international group for the psychology of mathematics education (PME-NA 32) (Vol. VI, Ch. 18: technology (pp. 1488–1496).

Roschelle, J. & Kaput, J. (1996). SimCalc Mathworlds for the mathematics of change. *Communications of the ACM, 39*, 97-99.

Schuck, S., (2016). Enhancing teacher education in primary mathematics with mobile technologies. *Australian Journal of Teacher Education, 41*(3), 126-139.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*, 404–411.

Star, J. R., & Newton, K. J. (2009). The nature and development of experts' strategy flexibility for solving equations. *ZDM Mathematics Education, 41*, 557–567.

Tatar, D., Roschelle, J., Knudsen, J., Shechtman, N., Kaput, J., & Hopkins, B. (2008). Scaling up innovative technology-based mathematics. *Journal of the Learning Sciences, 17*(2), 248-286.

Tchoshanov, M., Cruz, M. D., Huereca, K., Shakirova, K, Shakirov, L, & Ibragimova, E. N. (2017). Examination of lower secondary mathematics teachers' content knowledge and its connection to students' performance. *International Journal of Science and Mathematics Education, 15,* 683-702.

Yerushalmy, M., & Botzer, G. (2011). Guiding mathematical inquiry in mobile settings. In O. Zaslavsky, & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics* (pp. 191–207). New York: Springer.

Zazkis, R. & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8-13.